

Question 1

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Let $f(t) = e^{-t}$ and $g(t) = e^{-2t}$ for $t \geq 0$. A student claims that multiplying their transforms gives the transform of their pointwise product.

Tasks

1. Compute $c = f * g$ directly and prove $f * g = g * f$ by a change of variable.
2. Find the pointwise product $p = fg$. Compare $c(0)$, $p(0)$ and their transforms to assess the claim.
3. Derive the convolution theorem for these functions by changing variables in a double integral. State a real range of s that justifies the interchange of integrals.
4. Find the unique global maximum of c , its total area, and its exact real transform domain. Explain why the convolution initially vanishes even though both factors are positive at zero.

Question 1 – Solution

Strategy. Convolution integrates over a moving interval; pointwise multiplication does not.

Step 1: Integrate and reverse the factors. For $t \geq 0$,

$$c(t) = e^{-t} \int_0^t e^{-u} du = \boxed{e^{-t} - e^{-2t}}.$$

The substitution $v = t - u$ changes $\int_0^t f(t-u)g(u) du$ to $\int_0^t g(t-v)f(v) dv$, proving commutativity directly.

Step 2: Test the student's claim. Pointwise multiplication gives $p(t) = e^{-3t}$, so $p(0) = 1$ whereas $c(0) = 0$. Their transforms are

$$P(s) = \frac{1}{s+3}, \quad C(s) = \frac{1}{s+1} - \frac{1}{s+2} = \frac{1}{(s+1)(s+2)}.$$

The product $F(s)G(s)$ equals $C(s)$, not $P(s)$. For example, at $s = 0$ they have values $1/2$ and $1/3$ respectively.

Step 3: Derive the product formula. In the triangular region $0 \leq u \leq t < \infty$, put $v = t - u$. Its image is the quadrant $u, v \geq 0$, and its Jacobian has absolute value 1. For real $s > -1$, the resulting integrand has a finite absolute integral:

$$\begin{aligned} \mathcal{L}\{f * g\}(s) &= \int_0^\infty \int_0^t e^{-st} e^{-(t-u)} e^{-2u} du dt \\ &= \int_0^\infty \int_0^\infty e^{-(s+1)v} e^{-(s+2)u} du dv = F(s)G(s). \end{aligned}$$

Thus the interchange is justified by absolute integrability; it is not a formal rule for pointwise products.

Step 4: Locate the peak and inspect the tail. Since $c' = e^{-t}(2e^{-t} - 1)$, the derivative changes from positive to negative only at $t = \ln 2$. Consequently

$$\boxed{\max_{t \geq 0} c(t) = \frac{1}{4}, \quad \int_0^\infty c(t) dt = 1 - \frac{1}{2} = \frac{1}{2}.$$

The positive tail is asymptotic to e^{-t} , giving exact real domain $\boxed{s > -1}$; at $s = -1$ the transformed integrand tends to 1. At zero the integration interval has length zero. Finite endpoint values cannot produce a nonzero integral over that interval.