

Question 10

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

A leaky system receives a sequence of equal positive impulses:

$$y' + \kappa y = I \sum_{n=1}^{\infty} \delta(t - nT), \quad y(0) = 0, \quad I, \kappa, T > 0.$$

The first impulse is at T , not at zero. Use right-hand response values at each impulse.

Tasks

1. Derive $Y(s)$ by a geometric series and give the locally finite time-domain sum.
2. For $x_n = y(nT^+)$ and $x_0 = 0$, derive and solve the recurrence for post-impulse values.
3. Find the attracting periodic profile and the exact transient error. State its peak, pre-impulse trough and cycle mean, distinguishing a left limit from an attained value.
4. For $I = \kappa = 1$, classify all T for which the steady peak is at most 2 while the steady cycle mean is at least 1. State whether $y(t)$ has a single long-time limit.

Question 10 – Solution

Strategy. Each impulse adds I to the state, while the gap between impulses multiplies it by an exponential. The cycle map determines the limiting profile.

Step 1: Sum the transforms and invert. For $s > 0$, the impulse series converges geometrically:

$$(s + \kappa)Y = I \sum_{n=1}^{\infty} e^{-nTs} = \frac{Ie^{-Ts}}{1 - e^{-Ts}}, \quad Y = \frac{Ie^{-Ts}}{(s + \kappa)(1 - e^{-Ts})}.$$

Thus

$$y(t) = I \sum_{n=1}^{\infty} H_{nT}(t) e^{-\kappa(t-nT)}.$$

Only finitely many terms are active at each time. The formula gives zero before T , jumps of I , and $y' + \kappa y = 0$ between impulses.

Step 2: Solve the post-impulse recurrence. Put $q = e^{-\kappa T} \in (0, 1)$. Decay followed by a jump gives

$$x_n = qx_{n-1} + I, \quad x_n = \frac{I(1 - q^n)}{1 - q}, \quad n \geq 0.$$

The steady post-impulse level is $P = I/(1 - q)$.

Step 3: Construct the profile and compare. For $t = nT + \tau$, $0 \leq \tau < T$, define $y_{\text{per}}(t) = Pe^{-\kappa\tau}$. Then $y(t) = x_n e^{-\kappa\tau}$, including $n = 0$, and

$$y_{\text{per}}(t) - y(t) = Pe^{-\kappa t}.$$

The profile's peak P is attained just after each impulse. Its pre-impulse trough is the left limit qP ; under the right-hand convention this infimum is not attained within a cycle. Its mean is

$$\frac{1}{T} \int_0^T Pe^{-\kappa\tau} d\tau = \frac{I}{\kappa T}.$$

The jump $P - qP = I$ is preserved in the limit, not smoothed away.

Step 4: Design a feasible spacing and interpret. With $I = \kappa = 1$, the peak constraint gives $1/(1 - e^{-T}) \leq 2$, equivalent to $T \geq \ln 2$. The mean constraint gives $1/T \geq 1$, equivalent to $T \leq 1$. Therefore

$$\ln 2 \leq T \leq 1.$$

There is no single long-time limit: samples at nT tend to P , while samples at $nT + T/2$ tend to the different value $Pe^{-\kappa T/2}$. Both responses have exact real transform domain $s > 0$ because their persistent positive cycles prevent convergence at or below zero. The plot uses $I = \kappa = T = 1$ and marks the jump endpoints explicitly.

