

Question 8

Interpret the equation through ordinary motion away from the impulse and its jump conditions. For this question only, the initial-time transform convention is specified below; write $[v]_0 = v(0^+) - v(0^-)$ for a jump.

Use a *full-mass initial impulse* convention: the transform starts at 0^- , $\mathcal{L}\{\delta(t)\} = 1$, and

$$\mathcal{L}\{y'\} = sY - y(0^-), \quad \mathcal{L}\{y''\} = s^2Y - sy(0^-) - y'(0^-).$$

Consider

$$2y'' + 4y' + 2y = J\delta(t), \quad y(0^-) = 1, \quad y'(0^-) = 2.$$

These are pre-impact data, not post-impact data.

Tasks

1. Find $y(0^+)$ and $y'(0^+)$ by the impulse jump conditions.
2. Use the stated 0^- convention to find $Y(s)$ and the response for $t \geq 0$.
3. Reproduce the same response by solving a regular homogeneous IVP from 0^+ . Explain why keeping the impulse again would double-count it.
4. Classify every J for which the post-impact response is nonnegative and nonincreasing for all $t \geq 0$. Describe the boundary cases.

Question 8 – Solution

Strategy. An impulse at the initial time requires a declared convention. Either transform pre-impact data with the impulse or restart after the jump.

Step 1: Compute the post-impact data. There is no displacement jump under an ordinary force impulse. Integrating the equation across the origin gives $2[y']_0 = J$; the damping contribution is $4[y]_0 = 0$. Thus

$$y(0^+) = 1, \quad y'(0^+) = 2 + \frac{J}{2}.$$

The factor 2 is the mass and must not be omitted.

Step 2: Transform the pre-impact formulation. The stipulated derivative rules give

$$2(s^2Y - s - 2) + 4(sY - 1) + 2Y = J,$$

so

$$Y = \frac{2s + 8 + J}{2(s + 1)^2} = \frac{1}{s + 1} + \frac{3 + J/2}{(s + 1)^2}.$$

Writing $c = 3 + J/2$, the inverse is $y(t) = e^{-t}(1 + ct)$, $t \geq 0$. It has $y(0^+) = 1$ and $y'(0^+) = c - 1 = 2 + J/2$, exactly as the jump calculation.

Step 3: Check the equivalent restart. For $t > 0$ the equation is $y'' + 2y' + y = 0$. Using the post-impact data $1, 2 + J/2$ gives the critical solution $e^{-t}[1 + (3 + J/2)t]$ again. This restart has no remaining initial impulse: its effect is already encoded in the velocity. Inserting $J\delta(t)$ again with those data would add a second velocity change $J/2$.

Step 4: Classify positivity and monotonicity. Since the exponential is positive, nonnegativity for every $t \geq 0$ requires and is ensured by $c \geq 0$. Also

$$y' = e^{-t}(c - 1 - ct).$$

For $c \geq 0$, this is nonpositive for all $t \geq 0$ exactly when $c \leq 1$. Therefore

$$0 \leq c \leq 1 \iff -6 \leq J \leq -4.$$

At $J = -6$, $y = e^{-t}$; at $J = -4$, $y = (1 + t)e^{-t}$ starts horizontal and then decreases strictly. For $J < -6$ the response eventually becomes negative, and for $J > -4$ it initially increases. All these nonzero polynomial-exponential responses have exact real transform domain $s > -1$.

