

Question 7

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

An unstable second-order system is given one corrective impulse:

$$y'' - y = J\delta(t-a), \quad y(0) = 1, \quad y'(0) = 0, \quad a > 0.$$

Choose J to make the response decay to zero after the impulse. A realized impulse may differ by a constant error Δ .

Tasks

1. Derive $Y(s)$ and $y(t)$ for arbitrary J , including the initial response.
2. Find the unique decaying choice J_* , its explicit tail and the global maximum of the selected response.
3. Derive the response error for strength $J_* + \Delta$ and a sharp bound on $|\Delta|$ ensuring error at most $\epsilon > 0$ throughout $a \leq t \leq a + L$, where $L > 0$.
4. Compare exact real transform domains for the selected and perturbed strengths. Explain the effect of cancellation on the system's stability.

Question 7 – Solution

Strategy. Cancel the growing mode's coefficient, then inspect the remaining mode and the error produced by imperfect cancellation.

Step 1: Transform and invert. For sufficiently large s ,

$$(s^2 - 1)Y - s = Je^{-as}, \quad Y = \frac{s + Je^{-as}}{s^2 - 1}, \quad \boxed{y = \cosh t + JH_a(t) \sinh(t - a)}.$$

The added term starts at zero with derivative J , so it preserves displacement and produces the correct velocity jump. Away from a both terms solve the homogeneous equation.

Step 2: Select and interpret the decaying mode. After a , the e^t coefficient is $(1 + Je^{-a})/2$. Decay requires

$$\boxed{J_* = -e^a, \quad y_*(t) = \begin{cases} \cosh t, & 0 \leq t < a, \\ \cosh a e^{-(t-a)}, & t \geq a. \end{cases}}$$

The pre-impact velocity is $\sinh a$ and the post-impact velocity is $-\cosh a$; their difference is $-e^a$, as required. The response increases strictly before a and decreases strictly after it. Its unique global maximum is $\boxed{y_*(a) = \cosh a}$.

Step 3: Quantify the sensitivity. Linearity or direct subtraction gives

$$\boxed{y(t) - y_*(t) = \Delta H_a(t) \sinh(t - a)}.$$

Because $\sinh \tau$ increases strictly on $[0, L]$, the desired error bound is equivalent to

$$\boxed{|\Delta| \leq \frac{\epsilon}{\sinh L}}.$$

This is necessary as well as sufficient, and equality attains the bound at $a + L$. The selected solution still approaches zero only asymptotically; the impulse does not put it at the zero state.

Step 4: Check convergence and stability. The selected tail is a positive multiple of e^{-t} , so its exact real transform domain is $\boxed{s > -1}$. The apparent singularity at $s = 1$ in its transform is removable. For every $\Delta \neq 0$, the error contains a nonzero growing mode and the actual domain becomes $\boxed{s > 1}$. The differential equation's unstable homogeneous mode is unchanged. The plot uses $a = 1$, with a dashed curve for $\Delta = 0.05$.

