

Question 5

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a)dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

Two impulses act on an initially resting undamped oscillator:

$$y'' + \omega^2 y = I\delta(t-a) + K\delta(t-b), \quad y(0) = y'(0) = 0, \quad I, \omega > 0, \quad 0 < a < b.$$

Require complete rest after the second impulse.

Tasks

1. Find the transformed response and the time-domain solution.
2. Classify all delays $b-a$ and strengths K that meet the rest condition.
3. If K must be positive, find the earliest possible second impulse. Explain why the net impulse need not be zero.
4. Keep this earliest positive strength, but deliver it at $b+\eta$, where $|\eta| < \pi/\omega$. Find the residual amplitude and its leading behavior as $\eta \rightarrow 0$.

Question 5 – Solution

Strategy. The second kick must occur at a zero of displacement, then cancel the incoming velocity. Timing errors leave a measurable residual oscillation.

Step 1: Transform and invert. For $s > 0$,

$$Y = \frac{Ie^{-as} + Ke^{-bs}}{s^2 + \omega^2}, \quad \boxed{y = \frac{I}{\omega}H_a(t) \sin \omega(t-a) + \frac{K}{\omega}H_b(t) \sin \omega(t-b)}.$$

Each activated sine is zero initially, with derivative equal to its impulse strength. Thus y is continuous and $[y']_a = I$, $[y']_b = K$.

Step 2: Impose both state conditions. For $d = b - a$, the pre-impact state is

$$y(b-) = \frac{I}{\omega} \sin \omega d, \quad y'(b-) = I \cos \omega d.$$

Displacement cannot jump. Therefore complete rest requires and is ensured by

$$\boxed{d = \frac{n\pi}{\omega}, \quad K = (-1)^{n+1}I, \quad n = 1, 2, \dots}$$

After these jumps the homogeneous oscillator has zero data, so it remains zero. The corresponding output has compact support and a transform for every real s .

Step 3: Restrict the second impulse to be positive. Positive K requires odd n . The earliest choice is $\boxed{b = a + \pi/\omega, \quad K = I}$. The net applied impulse is $2I$, yet the oscillator stops: during the intervening motion the restoring force also changes momentum. Indeed $\int_a^b \omega^2 y dt = 2I$, balancing the applied impulses in the integrated equation. Ignoring this force would give an incorrect momentum test.

Step 4: Quantify a timing error. At the actual second kick the age is $\pi/\omega + \eta > 0$. The state immediately afterward is

$$y_+ = -\frac{I}{\omega} \sin \omega \eta, \quad v_+ = I(1 - \cos \omega \eta).$$

The subsequent homogeneous amplitude is

$$R = \sqrt{y_+^2 + (v_+/\omega)^2} = \boxed{\frac{2I}{\omega} |\sin(\omega\eta/2)|} = I|\eta| + O(|\eta|^3).$$

For every nonzero η in the stated interval this amplitude is positive, so the tail's exact real transform domain is $s > 0$. The plot compares perfect timing and $\eta = 0.25$ for $I = \omega = a = 1$.

