

Question 4

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

A critically damped system starts from rest and receives two impulses:

$$y'' + 2y' + y = \delta(t-1) + K\delta(t-b), \quad y(0) = y'(0) = 0, \quad b > 1.$$

You may choose b and K . The energy is $E = (y'^2 + y^2)/2$.

Tasks

1. Find $Y(s)$ and the response in terms of the age since each impulse.
2. Determine whether any finite $b > 1$ and K can make $y(t) = 0$ for every $t \geq b$.
3. For a fixed b , find the unique K minimizing the energy immediately after the second impulse, and find that minimum.
4. Derive the motion after this minimizing kick and explain why zero post-impact velocity does not imply complete rest.

Question 4 – Solution

Strategy. One impulse can adjust velocity only. Inspect the displacement at the chosen impact time before attempting to cancel the entire state.

Step 1: Find the impulse response. The zero-data transform is

$$Y = \frac{e^{-s} + Ke^{-bs}}{(s+1)^2}.$$

Since $\mathcal{L}\{te^{-t}\} = 1/(s+1)^2$, set $h(t) = te^{-t}$ for $t \geq 0$. Then

$$y = H_1(t)h(t-1) + KH_b(t)h(t-b).$$

Here $h(0) = 0$, $h'(0) = 1$, and $h'' + 2h' + h = 0$ for positive age. Thus displacement is continuous, and velocity jumps by 1 and K .

Step 2: Test the terminal displacement. Put $d = b - 1 > 0$. Immediately before the second kick,

$$y_b = de^{-d} > 0, \quad v_- = (1-d)e^{-d}.$$

The second impulse leaves y_b unchanged. A solution identically zero after b must have zero displacement there, so no finite $b > 1$ and K can produce complete rest. The strict positivity of $h(d)$ is the obstruction, not a shortage of algebraic manipulation.

Step 3: Minimize the post-impact energy. For fixed b ,

$$E_+(K) = \frac{1}{2}[(v_- + K)^2 + y_b^2].$$

This strictly convex quadratic has the unique minimizer and value

$$K_* = (d-1)e^{-d}, \quad E_{\min} = \frac{1}{2}d^2e^{-2d} > 0.$$

The minimizing kick makes $v_+ = 0$ but leaves positive stored spring energy.

Step 4: Follow the resulting motion. For $\tau = t - b \geq 0$, the critical homogeneous solution with state $(y_b, 0)$ is

$$y(b+\tau) = de^{-d}(1+\tau)e^{-\tau}.$$

Its derivative is $-de^{-d}\tau e^{-\tau}$: it starts horizontal and then moves strictly downward. Its acceleration at $\tau = 0+$ is $-y_b \neq 0$. Substitution verifies the equation and the terminal state. Thus the system eventually approaches rest, but is not at rest throughout the post-impact interval. The positive polynomial-exponential tail has transform domain $s > -1$.