

Question 3

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

An oscillator moving through equilibrium receives an impulse:

$$y'' + 4y = 6\delta(t - a), \quad y(0) = 1, \quad y'(0) = 0, \quad a = \frac{\pi}{4}.$$

For unit mass, use $E = (y'^2 + 4y^2)/2$. Ordinary impulses in a second-order equation keep displacement continuous but may jump velocity.

Tasks

1. Derive the velocity jump and find $Y(s)$ and $y(t)$.
2. Compute the states immediately before and after the impulse and simplify the subsequent motion.
3. Find the energy change and derive the general energy-change formula for an impulse of strength J at this same time.
4. For $J > 0$, classify when the impulse lowers the energy, preserves it, or raises it. Identify the unique strength that brings this oscillator completely to rest.

Question 3 – Solution

Strategy. The impulse changes velocity, not displacement. Its work depends on the velocity immediately before the kick as well as its strength.

Step 1: Jump, transform and invert. Integrating across a gives $[y']_a = 6$, because the integral of $4y$ tends to zero. With $[y]_a = 0$, the transformed equation is

$$Y = \frac{s + 6e^{-as}}{s^2 + 4}, \quad \boxed{y(t) = \cos 2t + 3H_a(t) \sin(2(t-a))}.$$

The new term vanishes at activation, while its derivative starts at 6. Both pieces satisfy $y'' + 4y = 0$ away from a and retain the initial data.

Step 2: Evaluate the state at impact. Since $2a = \pi/2$,

$$(y(a-), y'(a-)) = (0, -2), \quad (y(a+), y'(a+)) = \boxed{(0, 4)}.$$

For $t = a + \tau$, the pre-existing response is $-\sin 2\tau$, so the total post-impact motion is $\boxed{y(a + \tau) = 2 \sin 2\tau}$. Thus the impulse reverses the direction and doubles the amplitude.

Step 3: Compute the energy transfer. The energies are $E_- = 2$ and $E_+ = 8$, giving $\boxed{\Delta E = 6}$. For any impulse J , continuity of position and $v_+ = v_- + J$ imply

$$\Delta E = \frac{1}{2}[(v_- + J)^2 - v_-^2] = Jv_- + \frac{1}{2}J^2.$$

At this impact time $v_- = -2$, hence

$$\boxed{\Delta E = \frac{1}{2}J(J - 4), \quad E_+ = \frac{1}{2}(J - 2)^2}.$$

Step 4: Classify the positive kicks. For $0 < J < 4$ the energy decreases; at $J = 4$ it is unchanged; for $J > 4$ it increases. A positive impulse can remove energy when it opposes the incoming velocity. Complete rest requires both post-impact state coordinates to vanish. Position is already zero here, so the unique strength is $\boxed{J = 2}$, which makes velocity zero. A zero velocity kick at a nonzero displacement would not in general leave an oscillator at rest. The plot shows the stated $J = 6$ case, with continuous displacement and a corner.

