

Question 1

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

Let $a, J > 0$ and $\varepsilon > 0$. A finite pulse and its impulse idealization drive

$$y'_\varepsilon + 2y_\varepsilon = \frac{J}{\varepsilon}[H_a(t) - H_{a+\varepsilon}(t)], \quad y' + 2y = J\delta(t-a), \quad y_\varepsilon(0) = y(0) = 0.$$

Tasks

1. Prove that the finite pulse tends to $J\delta(t-a)$ when integrated against a continuous test function. Derive the limiting input transform.
2. Find the impulsive response, including its jump, and solve the finite-pulse IVP explicitly.
3. For $t \geq a + \varepsilon$, prove $0 \leq y_\varepsilon(t) - y(t) \leq J\varepsilon$.
4. Describe pointwise convergence, including the exceptional time a , and decide whether convergence can be uniform on an interval containing a in its interior.

Question 1 – Solution

Strategy. Preserve pulse area rather than height. Compare the two exact responses before making any claim about convergence.

Step 1: Test the pulse and transform. Its action on a continuous ϕ is

$$\frac{J}{\varepsilon} \int_a^{a+\varepsilon} \phi(t) dt \longrightarrow J\phi(a).$$

The error is at most $J \sup_{[a, a+\varepsilon]} |\phi(t) - \phi(a)|$, which tends to zero. Taking $\phi = e^{-st}$ gives $F_\varepsilon = J e^{-as}(1 - e^{-\varepsilon s})/(\varepsilon s) \rightarrow J e^{-as}$, with the removable value $F_\varepsilon(0) = J$.

Step 2: Solve both equations. The impulse gives $Y = J e^{-as}/(s+2)$, so $y = J H_a(t) e^{-2(t-a)}$ and $[y]_a = J$. The finite-pulse solution is continuous and equals

$$y_\varepsilon(t) = \begin{cases} 0, & t < a, \\ \frac{J}{2\varepsilon} (1 - e^{-2(t-a)}), & a \leq t < a + \varepsilon, \\ \frac{J}{2\varepsilon} (1 - e^{-2\varepsilon}) e^{-2(t-a-\varepsilon)}, & t \geq a + \varepsilon. \end{cases}$$

Each piece satisfies its ordinary equation, and the endpoints match.

Step 3: Bound the tail error. Put $\tau = t - a \geq \varepsilon$. The explicit tail can be rewritten as $y_\varepsilon = (J/\varepsilon) \int_0^\varepsilon e^{-2(\tau-u)} du$. Since $x \mapsto e^{-2x}$ is decreasing and has derivative magnitude at most 2 for $x \geq 0$,

$$0 \leq y_\varepsilon - y = \frac{J}{\varepsilon} \int_0^\varepsilon [e^{-2(\tau-u)} - e^{-2\tau}] du \leq \frac{J}{\varepsilon} \int_0^\varepsilon 2u du = J\varepsilon.$$

This representation follows directly from the finite-pulse formula.

Step 4: Distinguish the kinds of convergence. For each fixed $t \neq a$, $y_\varepsilon(t) \rightarrow y(t)$. At a , $y_\varepsilon(a) = 0$ but the right-hand representative has $y(a) = J$. Uniform convergence on an interval containing a is impossible: the approximants are continuous and the limit has a jump. Changing only $y(a)$ does not remove that jump or change its transform. The graph uses $a = J = 1$; open and filled dots mark the impulsive response.

