

Question 10

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A unit input is switched on and off repeatedly:

$$y' + y = f(t), \quad y(0) = 0, \quad f(t) = \begin{cases} 1, & 2k \leq t < 2k+1, \\ 0, & 2k+1 \leq t < 2k+2, \end{cases} \quad k = 0, 1, 2, \dots$$

Use only step functions and a geometric series; no periodic-transform formula is needed.

Tasks

1. Represent f by a locally finite sum of steps, derive $F(s)$ for $s > 0$, and obtain $Y(s)$ and a delayed-response series.
2. For $x_k = y(2k)$, derive and solve a recurrence for the samples.
3. Find the attracting two-periodic response explicitly on one cycle and prove the exact transient error for all $t \geq 0$.
4. Find its cycle minimum, maximum and mean. Does $y(t)$ have a limit? Find the earliest $T \geq 0$ for which the error from the periodic response is at most 0.01 for every $t \geq T$.

Question 10 – Solution

Strategy. The step series gives the transient solution, while the cycle map identifies a periodic state and its exact attraction rate.

Step 1: Transform the repeated switches. At each fixed time only finitely many terms are active:

$$f = \sum_{k=0}^{\infty} (H_{2k}(t) - H_{2k+1}(t)), \quad F = \frac{1 - e^{-s}}{s} \sum_{k=0}^{\infty} e^{-2ks} = \frac{1}{s(1 + e^{-s})}, \quad s > 0.$$

Integration of the nonnegative disjoint pulses justifies the sum. With $r(t) = 1 - e^{-t}$ for $t \geq 0$,

$$Y = \frac{1}{s(s+1)(1+e^{-s})}, \quad y = \sum_{k=0}^{\infty} [H_{2k}(t)r(t-2k) - H_{2k+1}(t)r(t-2k-1)].$$

This time series is locally finite; each activated response starts at zero and satisfies the required first-order equation on its active intervals.

Step 2: Solve the cycle recurrence. During the on-half, $y(2k+1) = 1 + (x_k - 1)e^{-1}$. During the off-half this value is multiplied by e^{-1} , so

$$x_{k+1} = e^{-2}x_k + e^{-1}(1 - e^{-1}), \quad \boxed{x_k = m(1 - e^{-2k}), \quad m = \frac{1}{e+1}},$$

using $x_0 = 0$. The cycle map has the unique fixed point m .

Step 3: Build the periodic state and exact error. Write $t = 2k + \tau$, $0 \leq \tau < 2$, and put $M = e/(e+1)$. Then

$$y_{\text{per}}(t) = \begin{cases} 1 - (1 - m)e^{-\tau}, & 0 \leq \tau < 1, \\ Me^{-(\tau-1)}, & 1 \leq \tau < 2. \end{cases}$$

The pieces match at M , and the value at the end of the cycle is m . Both solutions have the same forcing, so their difference satisfies $d' + d = 0$ with $d(0) = m$. Continuity propagates this identity across every switch:

$$\boxed{y_{\text{per}}(t) - y(t) = me^{-t}, \quad t \geq 0.}$$

Step 4: Interpret the limiting cycle and tolerance. The periodic minimum is m and maximum is M . Integrating its equation over a cycle gives $\int_0^2 y_{\text{per}} dt = 1$, so its mean is $1/2$. The even and odd integer samples of y tend to the different values m, M ; there is no single limit. The error decreases strictly, giving the sharp time

$$\boxed{T = \ln\left(\frac{100}{e+1}\right)}.$$

Both responses have exact real transform domain $s > 0$, since their positive persistent cycles prevent convergence at or below zero.

