

Question 9

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

Consider two qualitatively different input changes:

$$y'' + 3y' + 2y = H_1(t) + (t - 2)H_2(t), \quad y(0) = y'(0) = 0.$$

One claim says that the jump at 1 makes y jump; another says that because the ramp is continuous at 2, the solution must be smooth to every order there.

Tasks

1. Find $Y(s)$ and invert it explicitly using the step and ramp responses.
2. Verify the original IVP and determine the jumps in y, y', y'', y''' at 1 and 2.
3. Derive the leading small- τ behavior of each newly activated response. Relate the orders of onset to the jump calculations.
4. Evaluate both claims and explain whether a different assigned value of $H_a(a)$ changes the response.

Question 9 – Solution

Strategy. A second-order system smooths a step more than the input itself, but only by a finite number of derivatives. Check one-sided limits directly.

Step 1: Transform and invert. With zero data,

$$Y = \frac{e^{-s}}{s(s+1)(s+2)} + \frac{e^{-2s}}{s^2(s+1)(s+2)}.$$

Partial fractions give the two zero-start responses

$$r(\tau) = \frac{1}{2} - e^{-\tau} + \frac{1}{2}e^{-2\tau}, \quad p(\tau) = \frac{\tau}{2} - \frac{3}{4} + e^{-\tau} - \frac{1}{4}e^{-2\tau},$$

so $y = H_1(t)r(t-1) + H_2(t)p(t-2)$. Their transforms are the two unshifted rational factors displayed above.

Step 2: Verify the equation and all jumps. Direct differentiation gives $r'' + 3r' + 2r = 1$ and $p'' + 3p' + 2p = \tau$. Moreover $r(0) = r'(0) = p(0) = p'(0) = 0$, verifying the initial data and C^1 matching. For the next derivatives,

$$r''(0) = 1, \quad r'''(0) = -3, \quad p''(0) = 0, \quad p'''(0) = 1.$$

The previously active response is smooth at the next switch. Thus

switch	$[y]$	$[y']$	$[y'']$	$[y''']$
1	0	0	1	-3
2	0	0	0	1

These are jumps of one-sided ordinary derivatives, not assigned values at the switch.

Step 3: Relate onset orders to regularity. Taylor expansion of the newly added responses gives

$$r(\tau) = \frac{1}{2}\tau^2 - \frac{1}{2}\tau^3 + O(\tau^4), \quad p(\tau) = \frac{1}{6}\tau^3 + O(\tau^4), \quad \tau \downarrow 0.$$

Joining the first to zero preserves one derivative but creates a second- derivative jump. Joining the second to zero preserves two derivatives but creates a third-derivative jump. These leading coefficients reproduce the jump sizes above.

Step 4: Resolve both claims and the convention. The first claim is false: y and y' stay continuous at 1, while y'' jumps. The second is also false: y is C^2 near 2 but not C^3 there. Changing an input at one isolated time does not change its Laplace integral or the solution obtained by continuous state matching. At switches the equation is interpreted through its one-sided limits, not by demanding an unavailable classical derivative. Away from them the solution is smooth. The growing linear tail gives exact real transform domain $s > 0$.