

Question 8

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

An initially unexcited system is driven by one unknown positive step:

$$y' + 2y = AH_a(t), \quad y(0) = 0, \quad A > 0, \quad a > 0.$$

Exact measurements give $y(1) = 0$, $y(2) = 1$, and $y(3) = 2$. A fourth proposed measurement is $y(4) = 3$.

Tasks

1. Derive the transformed and time-domain response for arbitrary A, a .
2. Use positivity and the measurements to locate the switch interval, then determine A and a exactly.
3. Verify that the recovered parameters satisfy all three original measurements and are unique in this one-step family.
4. Test the fourth measurement and find the limiting response. Explain the scope of the identification.

Question 8 – Solution

Strategy. The zero measurement locates the delay. Once the input is on, equally spaced samples obey a simple exponential approach to the same level.

Step 1: Solve the parameterized IVP. For $s > 0$,

$$Y = \frac{Ae^{-as}}{s(s+2)}, \quad \boxed{y(t) = \frac{A}{2}H_a(t)(1 - e^{-2(t-a)}).$$

The response is zero through a , positive for $t > a$, and continuous at a . Its derivative jumps by A , as the equation requires.

Step 2: Locate and recover the switch. Since $A > 0$, the first two measurements imply $1 \leq a < 2$. Put $r = e^{-2}$ and $C = A/2$. On each full unit interval after activation,

$$y(t+1) = ry(t) + C(1-r).$$

Thus $2 = r + C(1-r)$, and

$$\boxed{A = \frac{2(2-r)}{1-r}}.$$

Using $y(2) = 1$ gives $e^{-2(2-a)} = 1 - 1/C = 1/(2-r)$, hence

$$\boxed{a = 2 - \frac{1}{2} \ln(2 - e^{-2})}.$$

Step 3: Verify admissibility and uniqueness. Because $1 < 2 - r < 2 < e^2$, one has $0 < \ln(2 - r) < 2$, so $1 < a < 2$. Also $C = (2 - r)/(1 - r) > 0$. The recovered values give $y(1) = 0$ and

$$y(2) = C \left(1 - \frac{1}{2-r} \right) = 1, \quad y(3) = r + C(1-r) = 2.$$

The sample recurrence fixes C uniquely. Then the exponential expression for $y(2)$ is strictly monotone in a , fixing the delay uniquely. Direct substitution into the response verifies the IVP on both sides of the switch.

Step 4: Test the extra datum and interpret. The same recurrence predicts

$$\boxed{y(4) = 2r + C(1-r) = 2 + e^{-2} < 3.}$$

The fourth datum is inconsistent with the stated model. The limiting level is $\boxed{C = (2 - e^{-2})/(1 - e^{-2})}$, approached from below after activation. Its positive constant tail gives exact real transform domain $s > 0$. Uniqueness here concerns the specified single positive step with known system coefficient; finitely many samples do not identify an arbitrary forcing history.