

Question 7

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

An unstable first-order system is to be stopped by a two-stage input:

$$y' - y = f_B(t), \quad y(0) = 1, \quad f_B(t) = \begin{cases} -1, & 0 \leq t < 1, \\ B, & 1 \leq t < 2, \\ 0, & t \geq 2. \end{cases}$$

Choose $B = B_*$ so that $y(t) = 0$ for all $t \geq 2$. A realized level may instead be $B_* + \delta$, where δ is a constant error.

Tasks

1. Find $Y(s)$ and invert it using the unstable system's step response.
2. Determine B_* and the resulting piecewise output. Verify the stopping condition.
3. Derive the exact output error for $t \geq 2$ and the sharp bound on $|\delta|$ that ensures $|y(t)| \leq 0.01$ for every $2 \leq t \leq 5$.
4. Compare the exact real transform domains for perfect and imperfect levels. Explain why exact cancellation does not make the system stable.

Question 7 – Solution

Strategy. A finite input can cancel one unstable trajectory exactly. Keep the cancellation coefficient explicit to measure its sensitivity.

Step 1: Transform and invert. Here $f_B = -1 + (B + 1)H_1(t) - BH_2(t)$ and

$$Y = \frac{1 + [-1 + (B + 1)e^{-s} - Be^{-2s}]/s}{s - 1}, \quad s > 1.$$

The zero-start response to a unit step for $y' - y = 1$ is $r(t) = e^t - 1$. Adding the initial response gives

$$y = e^t - r(t) + (B + 1)H_1(t)r(t - 1) - BH_2(t)r(t - 2).$$

This formula satisfies $y(0) = 1$ and is continuous at both switches.

Step 2: Select the exact cancellation. Before 1, $y = 1$. On $[1, 2]$, $y = (B + 1)e^{t-1} - B$, so $y(2) = e + B(e - 1)$. Consequently

$$B_* = -\frac{e}{e - 1}, \quad y_*(t) = \begin{cases} 1, & 0 \leq t < 1, \\ (e - e^{t-1})/(e - 1), & 1 \leq t < 2, \\ 0, & t \geq 2. \end{cases}$$

Direct substitution gives $y'_* - y_* = -1, B_*, 0$ on the three intervals, and the values match at 1 and 2. The zero terminal state remains zero under the homogeneous equation after 2.

Step 3: Quantify the error sharply. The error is zero before 1, then equals $\delta(e^{t-1} - 1)$ until 2. After 2 the homogeneous equation amplifies its terminal value:

$$y(t) - y_*(t) = \delta(e - 1)e^{t-2}, \quad t \geq 2.$$

Its magnitude increases on $[2, 5]$ when $\delta \neq 0$. Thus the required necessary and sufficient tolerance is

$$|\delta| \leq \frac{0.01 e^{-3}}{e - 1}.$$

Equality attains the limit at $t = 5$.

Step 4: Distinguish trajectory cancellation from stability. The perfect output has compact support and a transform for every real s . For any $\delta \neq 0$, a nonzero e^t tail remains and the exact real domain is $s > 1$. At the perfect setting the apparent singularity at $s = 1$ is removable, as is the $s = 0$ expression after cancellations. Small level errors eventually grow without bound; selecting one compact response does not change the unstable homogeneous equation.

