

Question 6

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

Choose two constant force levels to bring an oscillator exactly to rest:

$$y'' + y = f(t), \quad y(0) = 1, \quad y'(0) = 0, \quad f(t) = \begin{cases} A, & 0 \leq t < a, \\ B, & a \leq t < T, \\ 0, & t \geq T, \end{cases} \quad a = \frac{\pi}{4}, \quad T = \frac{\pi}{2}.$$

Require $y(t) = 0$ for every $t \geq T$. The levels A, B may have either sign.

Tasks

1. Express f using steps and derive $Y(s)$, including the initial response.
2. Invert $Y(s)$ and derive the two linear equations that the terminal conditions impose on A, B .
3. Solve for both force levels and prove that they give a unique motion which remains at rest after T .
4. Decide whether nonnegative levels could meet the target. Describe the smoothness at both switches and the exact real transform domain of the designed output.

Question 6 – Solution

Strategy. Exact rest requires two terminal state conditions. Use the step response to convert them into two equations for the two force levels.

Step 1: Transform with the initial displacement. The level changes give $f = A + (B - A)H_a(t) - BH_T(t)$, so

$$(s^2 + 1)Y - s = \frac{A + (B - A)e^{-as} - Be^{-Ts}}{s}.$$

Writing $r(t) = 1 - \cos t$, the inverse is

$$y = \cos t + Ar(t) + (B - A)H_a(t)r(t - a) - BH_T(t)r(t - T).$$

The unforced term $\cos t$ supplies the initial data. Since $r(0) = r'(0) = 0$, each added response preserves y, y' at activation, and $r'' + r = 1$ verifies the forcing on each open interval.

Step 2: Impose both terminal conditions. Set $c = \cos a = \sin a = 1/\sqrt{2}$. At $T = \pi/2$ the last response and its derivative still vanish. The terminal values are

$$y(T) = cA + (1 - c)B, \quad y'(T) = -1 + (1 - c)A + cB.$$

Thus rest requires

$$cA + (1 - c)B = 0, \quad (1 - c)A + cB = 1.$$

Step 3: Solve and prove complete rest. The determinant is $c^2 - (1 - c)^2 = 2c - 1 = \sqrt{2} - 1 > 0$. Consequently

$$A = -\frac{1 - c}{2c - 1} = \boxed{-\frac{1}{\sqrt{2}}}, \quad B = \frac{c}{2c - 1} = \boxed{1 + \frac{1}{\sqrt{2}}}.$$

These are the unique levels in the stated two-stage family. The formula satisfies the original IVP and has zero displacement and velocity at T . For $t > T$ the forcing is zero, so uniqueness of the homogeneous oscillator with these terminal data implies $y(t) = 0$ there. Matching gives a unique solution on the entire half-line.

Step 4: Check feasibility and regularity. Both weights in $cA + (1 - c)B = 0$ are positive. If $A, B \geq 0$, that equation forces $A = B = 0$, contradicting the velocity equation. A negative first level is therefore necessary within this family. The motion is C^1 , while the acceleration jumps are

$$[y'']_a = B - A = 1 + \sqrt{2}, \quad [y'']_T = -B = -1 - \frac{1}{\sqrt{2}}.$$

After T the output is identically zero, so its transform exists for $\boxed{\text{every real } s}$, despite the apparent denominators in Y .