

Question 5

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A resonant input is windowed without restarting its phase:

$$y'' + y = \sin t [H_a(t) - H_b(t)], \quad y(0) = y'(0) = 0, \quad a = \frac{\pi}{2}, \quad b = \frac{3\pi}{2}.$$

Tasks

1. Rewrite the forcing in delayed coordinates and derive its transform. Pay particular attention to the sign at the second switch.
2. Find and invert $Y(s)$, deriving the required repeated-quadratic transform pair.
3. Find the exact motion after b and its displacement, velocity and energy at shutoff.
4. Determine the acceleration jumps and the exact real transform domain. Explain why a zero displacement at shutoff does not mean rest.

Question 5 – Solution

Strategy. The two endpoints have opposite sine phases. Rewriting both phases first prevents a sign error in the delayed inverse.

Step 1: Shift the phases. For delayed time τ , $\sin(\tau + a) = \cos \tau$ and $\sin(\tau + b) = -\cos \tau$. Therefore

$$f = H_a(t) \cos(t - a) + H_b(t) \cos(t - b), \quad F = \frac{s(e^{-as} + e^{-bs})}{s^2 + 1}.$$

The plus sign at b cancels the continuing negative-phase input; it does not add a second physical forcing window.

Step 2: Derive and invert the response pair. The zero-data equation gives

$$Y = \frac{s(e^{-as} + e^{-bs})}{(s^2 + 1)^2}, \quad s > 0.$$

Differentiating $\mathcal{L}\{\sin t\} = 1/(s^2 + 1)$ with respect to s gives $\mathcal{L}\{t \sin t/2\} = s/(s^2 + 1)^2$. With $q(\tau) = \tau \sin \tau/2$,

$$\boxed{y = H_a(t)q(t - a) + H_b(t)q(t - b).}$$

Here $q(0) = q'(0) = 0$ and $q'' + q = \cos \tau$, directly verifying the equation, the zero initial data and both matching conditions.

Step 3: Evaluate the residual motion. For $t = b + v$, $v \geq 0$, the two ages are $v + \pi$ and v . Hence

$$y(b + v) = q(v + \pi) + q(v) = \boxed{-\frac{\pi}{2} \sin v}.$$

In particular $y(b) = 0$, $y'(b) = -\pi/2$, and the post-shutoff energy is

$$\boxed{E = \frac{1}{2}(y'^2 + y^2) = \frac{\pi^2}{8}}.$$

The pulse ends precisely at a zero crossing with nonzero velocity, so the system continues oscillating with amplitude $\pi/2$.

Step 4: Inspect the switches and convergence. The input jumps from 0 to 1 at a , and from -1 to 0 at b . Continuity of y gives $\boxed{[y'']_a = [y'']_b = 1}$. There are no jumps in y or y' . The nonzero sinusoidal tail has exact real transform domain $\boxed{s > 0}$: its undamped primitive fails to converge at zero, and exponential growth of fixed-sign tail integrals rules out negative s .

