

Question 3

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

An undamped oscillator receives a rectangular force of selectable duration:

$$y'' + \omega^2 y = F[1 - H_T(t)], \quad y(0) = y'(0) = 0, \quad F > 0, \omega > 0, T > 0.$$

Use the energy $E = (y'^2 + \omega^2 y^2)/2$.

Tasks

1. Derive $Y(s)$ and the delayed step-response formula for y .
2. Find the displacement and velocity at shutoff, and the residual oscillation amplitude.
3. Classify every duration T for which the system remains at rest after shutoff; explain why $y(T) = 0$ alone is generally insufficient.
4. Compute the residual energy and classify the exact real transform domain, including the rest-producing durations.

Question 3 – Solution

Strategy. Removing the force subtracts a delayed step response. The two surviving state coordinates determine whether motion persists.

Step 1: Transform and invert. For $s > 0$,

$$Y = \frac{F(1 - e^{-Ts})}{s(s^2 + \omega^2)}, \quad \boxed{y = \frac{F}{\omega^2} [1 - \cos \omega t - H_T(t)(1 - \cos \omega(t - T))].}$$

Before T this has zero initial data and satisfies the forced equation. The subtracted response and its first derivative vanish at activation, so y, y' match at T ; after T the equation is homogeneous.

Step 2: Recover the shutoff state and amplitude. The matching state is

$$y(T) = \frac{F}{\omega^2}(1 - \cos \omega T), \quad y'(T) = \frac{F}{\omega} \sin \omega T.$$

For $t \geq T$, subtraction and the cosine difference identity give

$$y(t) = \frac{2F}{\omega^2} \sin(\omega T/2) \sin(\omega(t - T/2)).$$

Its amplitude is therefore $\boxed{R = 2F|\sin(\omega T/2)|/\omega^2}.$

Step 3: Classify complete return to rest. A homogeneous oscillator remains zero exactly when both state coordinates are zero. Here that happens precisely for

$$\boxed{T = \frac{2\pi n}{\omega}, \quad n = 1, 2, \dots}$$

In this particular pulse family, $y(T) = 0$ forces $\cos \omega T = 1$ and hence also $y'(T) = 0$. For a general forcing history, zero displacement alone is insufficient: a nonzero velocity launches a new oscillation.

Step 4: Compute energy and convergence. After shutoff,

$$\boxed{E = \frac{1}{2}\omega^2 R^2 = \frac{2F^2}{\omega^2} \sin^2(\omega T/2).}$$

If $R > 0$, the oscillatory tail has an ordinary Laplace transform exactly for $s > 0$: at zero its primitive does not converge, and negative s gives growing integrals over fixed sign intervals. If $R = 0$, the output has compact support, so its transform exists for every real s . Apparent poles in the displayed formula then cancel. The graph compares $F = \omega = 1$ at two durations.

