

Question 2

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A ramp is switched on without resetting its clock, then switched off:

$$y' + y = f(t), \quad y(0) = 0, \quad f(t) = \begin{cases} t, & 1 \leq t < 3, \\ 0, & \text{otherwise.} \end{cases}$$

A proposed input transform is $(e^{-s} - e^{-3s})/s^2$.

Tasks

1. Explain the error in the proposed transform and derive the correct $F(s)$.
2. Find $Y(s)$ and invert it, rewriting each active input in its own delayed time.
3. Give the piecewise solution, its maximum, and the two jumps in y' .
4. Compute $\int_0^\infty y(t) dt$ directly and from the differential equation. Explain the apparent singularities in the transform formula.

Question 2 – Solution

Strategy. A window does not reset the ramp's clock. At each endpoint rewrite t as delayed time plus the endpoint.

Step 1: Shift the whole input. The input is $H_1(t)[(t-1)+1] - H_3(t)[(t-3)+3]$, so

$$F = e^{-s} \left(\frac{1}{s^2} + \frac{1}{s} \right) - e^{-3s} \left(\frac{1}{s^2} + \frac{3}{s} \right).$$

The proposed expression instead transforms $H_1(t)(t-1) - H_3(t)(t-3)$, which has a constant tail of 2 after 3. It describes neither the given ramp window nor its shutoff.

Step 2: Invert the response. The zero datum gives $Y = F/(s+1)$. For a delayed input $\tau + c$, the zero-start response solves $r'_c + r_c = \tau + c$, and is

$$r_c(\tau) = \tau + c - 1 + (1 - c)e^{-\tau}.$$

Its transform is $(1/s^2 + c/s)/(s+1)$, as direct transformation confirms. In particular $r_1 = \tau$ and $r_3 = \tau + 2 - 2e^{-\tau}$. Thus

$$y = H_1(t)(t-1) - H_3(t)[(t-3) + 2 - 2e^{-(t-3)}].$$

Step 3: Read the intervals and switches. Simplifying after the second switch gives

$$y(t) = \begin{cases} 0, & 0 \leq t < 1, \\ t-1, & 1 \leq t < 3, \\ 2e^{-(t-3)}, & t \geq 3. \end{cases}$$

The pieces meet at 0 and 2, and satisfy $y' + y = 0, t, 0$. Their slopes are $0, 1, -2e^{-(t-3)}$, so $[y']_1 = 1$ and $[y']_3 = -3$. The unique global maximum is $y(3) = 2$, by increase during the window and strict decay afterward.

Step 4: Check the total response. Directly,

$$\int_0^\infty y \, dt = \int_1^3 (t-1) \, dt + \int_3^\infty 2e^{-(t-3)} \, dt = 2 + 2 = 4.$$

Integrating the equation gives the same answer because $y(\infty) - y(0) = 0$ and $\int_1^3 t \, dt = 4$. The compact input has a transform for every real s , with its apparent $s = 0$ poles canceling. The output's nonzero exponential tail gives exact domain $s > -1$, and the removable value $Y(0) = 4$ agrees with the area.