

Question 9

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

An unknown parameter $\alpha > 0$ appears in

$$(1+t)y' + \alpha y = 0, \quad y(0) = 1.$$

An exact measurement gives $\int_0^\infty y(t) dt = 1/2$. A second proposed measurement is $\int_0^\infty ty(t) dt = 1/3$.

Tasks

1. Derive the differential equation for Y and solve it as a convergent parameter integral for $s > 0$.
2. Identify and verify the time solution for general $\alpha > 0$.
3. Determine which parameters admit a finite area, then recover α from the first measurement.
4. Test the proposed first moment for consistency, and state the exact real transform domain of the recovered solution.

Question 9 – Solution

Strategy. Solve the parameter family before fitting measurements. A boundary transform value is meaningful only when the actual time integral converges.

Step 1: Derive and solve the transform equation. Using $\mathcal{L}\{y'\} = sY - 1$ and $\mathcal{L}\{ty'\} = -Y - sY'$ gives

$$-sY' + (s + \alpha - 1)Y = 1.$$

The integrating factor for the divided equation is $e^{-s}s^{1-\alpha}$, so its product with Y has derivative $-e^{-s}s^{-\alpha}$. The transform bound at infinity selects

$$Y(s) = e^s s^{\alpha-1} \int_s^\infty e^{-u} u^{-\alpha} du, \quad s > 0.$$

The alternative homogeneous term $Ce^s s^{\alpha-1}$ violates decay as $s \rightarrow \infty$.

Step 2: Verify the parameter family in time. Separation gives $y(t) = (1+t)^{-\alpha}$. Its derivative is $-\alpha(1+t)^{-\alpha-1}$, so the equation and initial value hold. The coefficient $1+t$ is nonzero on $t \geq 0$, establishing regular-IVP uniqueness. Substitution $u = s(1+t)$ in the forward integral gives exactly the parameter expression above, verifying its inversion.

Step 3: Use the convergent area to identify the parameter. The positive power tail has finite area exactly when $\alpha > 1$, and then

$$\int_0^\infty (1+t)^{-\alpha} dt = \frac{1}{\alpha-1}.$$

Equating this to $1/2$ gives the unique value $\alpha = 3$. This uses convergence of the integral, not a formal substitution into an expression initially derived only for $s > 0$.

Step 4: Test the second datum rather than refit. A finite first moment requires $\alpha > 2$. Substituting $x = 1+t$ gives

$$\int_0^\infty \frac{t}{(1+t)^\alpha} dt = \frac{1}{\alpha-2} - \frac{1}{\alpha-1} = \frac{1}{(\alpha-1)(\alpha-2)}.$$

At $\alpha = 3$, it is $1/2$, not $1/3$, so the proposed data are inconsistent with this family. The recovered positive tail is $(1+t)^{-3}$; its exact transform domain is $[0, \infty)$. Positive s gives exponential damping, zero gives the finite area $1/2$, and negative s makes the weighted power tail diverge.