

Question 8

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Classify the data for which a twice continuously differentiable solution on $[0, \infty)$ can satisfy

$$t^2 y'' + 2ty' - 2y = 0, \quad y(0) = a, \quad y'(0) = b.$$

For a smooth exponential-order function with $a = 0$, you may use $\lim_{s \rightarrow \infty} s^2 Y(s) = b$.

Tasks

1. Find the compatibility condition at zero and solve the time equation for $t > 0$ using trial powers t^m .
2. Derive the second-order differential equation for Y , accounting for the effect of two parameter derivatives on the initial terms.
3. Solve the transformed equation using trial powers s^m . Determine what the condition $Y \rightarrow 0$ fixes and what initial information is still needed.
4. Classify all admissible pairs (a, b) , verify the resulting solutions, and state their exact transform domains. Is specifying only $y(0)$ enough for uniqueness?

Question 8 – Solution

Strategy. Singular coefficients can erase initial data from the transformed equation. Recover admissibility and the remaining amplitude separately.

Step 1: Check compatibility and time powers. At zero, the equation requires $\boxed{a = 0}$. For $t > 0$, trial powers give $m(m-1) + 2m - 2 = (m-1)(m+2) = 0$. The two independent solutions are t and t^{-2} , so $y = Ct + Dt^{-2}$. They span the solutions on $(0, \infty)$ because the normalized equation is regular there. Continuity at zero forces $D = 0$; the initial slope then sets $C = b$.

Step 2: Transform the polynomial coefficients. For sufficiently large s ,

$$\mathcal{L}\{t^2 y''\} = \frac{d^2}{ds^2}[s^2 Y - sa - b] = 2Y + 4sY' + s^2 Y'',$$

while $2\mathcal{L}\{ty'\} = -2Y - 2sY'$. Adding $-2Y$ gives

$$\boxed{s^2 Y'' + 2sY' - 2Y = 0.}$$

Both a and b vanished under differentiation; their disappearance is not permission to ignore the time-domain data.

Step 3: Select the transform and its amplitude. Trial powers of s have exponents one and -2 , yielding $Y = As + B/s^2$ on $s > 0$. A genuine transform of an exponential-order continuous function tends to zero at large s , so $A = 0$. This still leaves B arbitrary. The supplied initial-slope limit gives $\boxed{B = b}$ and hence $Y = b/s^2$. The initial-value limit $sY \rightarrow 0$ also agrees with the necessary datum $a = 0$.

Step 4: Verify and classify uniqueness. All and only the data $\boxed{a = 0, b \in \mathbb{R}}$ are admissible, with unique solution $\boxed{y = bt}$. Substitution gives $0 + 2tb - 2bt = 0$, and both data hold. For $b \neq 0$, the exact domain is $s > 0$; the signed linear tail diverges at or below zero. For $b = 0$, the zero function has transform zero for every real s . Specifying only $y(0) = 0$ leaves the entire family bt , so it does not determine a unique solution at this singular point.