

Question 7

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Consider the local IVP

$$(t-1)y' + y = 0, \quad y(0) = 1.$$

For $0 < T < 1$, define the finite transform $F_T(s) = \int_0^T e^{-st}y(t) dt$.

Tasks

1. Solve the IVP up to its first singularity and decide whether a global classical solution on $[0, \infty)$ exists.
2. Formally applying the infinite-transform rules gives $Y' + Y = 1/s$. Identify why this equation cannot be used as an ordinary transform solution of the IVP.
3. Derive the correct differential equation for F_T by retaining the upper boundary term during integration by parts.
4. Check that finite-transform equation directly, including its value at $s = 0$. Explain why a finite cutoff repairs the calculation but does not remove the singularity.

Question 7 – Solution

Strategy. An interior zero of the leading coefficient can stop the classical solution. A finite transform before that point must retain its endpoint term.

Step 1: Locate the obstruction in time. Since $[(t-1)y]' = 0$, the datum gives $(t-1)y = -1$. Thus

$$y(t) = \frac{1}{1-t}, \quad 0 \leq t < 1.$$

This diverges as $t \uparrow 1$ and admits no continuous classical continuation through one. There is no global classical IVP solution on the full half-line. The integral up to one already diverges for every real s , because its positive weight stays bounded below near one while $1/(1-t)$ has a logarithmic divergence.

Step 2: Diagnose the formal infinite equation. The formal substitutions $\mathcal{L}\{ty'\} = -Y - sY'$ and $\mathcal{L}\{y'\} = sY - 1$ would give $Y' + Y = 1/s$. But the required infinite time integrals do not exist and no global classical solution satisfies the starting hypotheses. Solving this formal equation cannot provide an ordinary transform of the local IVP solution.

Step 3: Retain the actual cutoff boundary. For $T < 1$, integration by parts gives

$$0 = [e^{-st}(t-1)y(t)]_0^T + s \int_0^T e^{-st}(t-1)y(t) dt.$$

The first term is $1 - e^{-sT}$, using $(T-1)y(T) = -1$. Since $\int_0^T te^{-st}y = -F'_T$, the result is

$$F'_T(s) + F_T(s) = \frac{1 - e^{-sT}}{s} \quad (s \neq 0).$$

Discarding the upper endpoint would lose the term $-e^{-sT}/s$.

Step 4: Verify the finite identity and its limits. Direct differentiation on the compact interval gives

$$F'_T + F_T = \int_0^T (1-t)e^{-st}y(t) dt = \int_0^T e^{-st} dt.$$

This is the stated expression for $s \neq 0$ and equals $\boxed{T}$ at $s = 0$. Also $F_T(0) = -\ln(1-T)$ and $F'_T(0) = T + \ln(1-T)$, whose sum is T . Every finite F_T exists for all real s , but it diverges as $T \uparrow 1$ for each fixed s . A valid finite-window calculation does not supply a missing continuation through the pole. The plot stops short of the singular time.

