

Question 6

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Solve the regular second-order IVP

$$(1+t)y'' + y' = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

A convergent parameter integral may be retained in the formula for Y .

Tasks

1. Derive the differential equation for Y , including all initial contributions.
2. Solve it with an integrating factor and explain the condition that removes its homogeneous term.
3. Identify and verify the time solution, then check the parameter-integral transform directly.
4. Prove $0 < Y(s) < 1/s^2$ for $s > 0$ and determine the exact real convergence interval. Explain why a growing time solution can still be transformable.

Question 6 – Solution

Strategy. Keep the initial derivative term from the unmultiplied second derivative. The coefficient $1 + t$ never vanishes on the time domain.

Step 1: Transform with both data. The three contributions are

$$\mathcal{L}\{ty''\} = -2sY - s^2Y', \quad \mathcal{L}\{y''\} = s^2Y - 1, \quad \mathcal{L}\{y'\} = sY.$$

Hence $-s^2Y' + (s^2 - s)Y = 1$, or

$$Y' - (1 - 1/s)Y = -1/s^2, \quad s > 0.$$

Step 2: Select the parameter integral. The integrating factor is se^{-s} , so $(se^{-s}Y)' = -e^{-s}/s$. An exponential-order continuous time solution has $Y = O(1/s)$ for large s , and thus $se^{-s}Y \rightarrow 0$. Therefore

$$Y(s) = \frac{e^s}{s} \int_s^\infty \frac{e^{-u}}{u} du.$$

An additional term Ce^s/s violates the large-parameter bound unless $C = 0$. Existence and the growth condition are verified by the recovered solution.

Step 3: Identify and check the inverse. In time, the left side is $[(1+t)y']'$. Integrating with the data gives $y' = 1/(1+t)$ and

$$y(t) = \ln(1+t).$$

It has $y(0) = 0$, $y'(0) = 1$, $y'' = -1/(1+t)^2$, and zero equation residual. The coefficient is regular, so uniqueness applies. Integrating its transform by parts gives

$$Y(s) = \frac{1}{s} \int_0^\infty \frac{e^{-st}}{1+t} dt = \frac{e^s}{s} \int_s^\infty \frac{e^{-u}}{u} du,$$

where the upper boundary vanishes for $s > 0$.

Step 4: Compare growth rates and domains. For $t > 0$, $0 < \ln(1+t) < t$. Multiplying by the positive kernel and integrating gives

$$0 < Y(s) < \frac{1}{s^2}, \quad s > 0.$$

The exact domain is $s > 0$: logarithmic growth is dominated by the exponential kernel above zero, whereas its positive time integral diverges at zero and for negative parameters. Unboundedness by itself does not prevent a Laplace transform; growth relative to the kernel is what matters.