

Question 4

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Consider a singular initial point:

$$ty' + 2y = e^{-t}, \quad y(0) = \frac{1}{2}.$$

Seek a solution continuously differentiable on $[0, \infty)$ and satisfying the equation also at zero.

Tasks

1. Check compatibility at zero and determine whether boundedness near zero leaves a free integration constant in the time equation.
2. Derive and solve the equation for Y , selecting a branch that tends to zero at large positive s .
3. Identify and verify the inverse, including its limiting value and derivative at zero.
4. Find the full real convergence set and $Y(0)$. Explain why a finite transform value at zero need not imply a finite derivative there.

Question 4 – Solution

Strategy. A vanishing leading coefficient imposes compatibility. Boundedness at the singular endpoint can replace a free constant that would exist away from it.

Step 1: Inspect the singular endpoint. At zero, the equation requires $2y(0) = 1$, so the given datum is compatible. For $t > 0$, multiplying by t gives $(t^2y)' = te^{-t}$. Its general integral has an additive constant in t^2y . Boundedness of y forces that constant to be zero, hence

$$y(t) = \frac{1 - (1+t)e^{-t}}{t^2} \quad (t > 0).$$

This proves uniqueness among bounded solutions near zero, without invoking a regular-IVP theorem at a singular point.

Step 2: Solve in transform space. The full rule $\mathcal{L}\{ty'\} = -Y - sY'$ gives $Y - sY' = 1/(s+1)$. Therefore

$$(Y/s)' = -\frac{1}{s^2(s+1)}, \quad Y = s \int_s^\infty \frac{du}{u^2(u+1)} = \boxed{1 - s \ln \frac{s+1}{s}}, \quad s > 0.$$

The omitted homogeneous term is Cs and cannot tend to zero. The inverse below verifies boundedness and transform existence.

Step 3: Verify the endpoint and forward transform. A nonsingular representation is $y(t) = \int_0^1 ue^{-ut} du$, including at zero. Thus $y(0) = 1/2$, $y'(0) = -1/3$, and differentiation verifies the original equation (or use $(t^2y)' = te^{-t}$ for $t > 0$ and compatibility at zero). For $s > 0$, integrating first in time gives

$$Y(s) = \int_0^1 \frac{u}{s+u} du = 1 - s \ln \frac{s+1}{s}.$$

The finite absolute double integral justifies interchange.

Step 4: Check the boundary rather than just the formula. The function is positive and $y(t) \sim 1/t^2$. Therefore the exact real convergence set is $\boxed{[0, \infty)}$, with divergence for negative s . At zero the same

nonnegative double integral gives $\boxed{Y(0) = \int_0^1 1 du = 1}$. But $ty(t) \sim 1/t$, so its first moment diverges.

Indeed $Y'(s) = -\ln[(s+1)/s] + 1/(s+1) \rightarrow -\infty$ as $s \downarrow 0$. A finite boundary area does not imply a finite boundary moment.

