

Question 3

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Investigate the validity of the Laplace method for

$$y' - 2ty = 0, \quad y(0) = 1, \quad t \geq 0.$$

Do not assume that every global smooth IVP solution has an ordinary Laplace transform.

Tasks

1. Solve the time-domain IVP and determine whether its ordinary transform exists for any real s .
2. Formally apply the transform rules and solve the resulting equation for a candidate Y , without yet claiming it is a transform.
3. Show that every member of this formal family satisfies $Y(s) \rightarrow 0$ and $sY(s) \rightarrow 1$ as $s \rightarrow \infty$.
4. Explain why neither those limits nor solving the formal equation validates the calculation. Identify the failed hypothesis and contrast it with the sign-reversed damping equation.

Question 3 – Solution

Strategy. Check the actual time solution first. Necessary transform limits can hold for functions of s that are not transforms of the IVP solution.

Step 1: Establish nonexistence of an ordinary transform. Separation and the initial datum give $y = e^{t^2}$, which is smooth and global. For any fixed real s , however, $t^2 - st \geq t^2/2$ for sufficiently large t . Thus

$$\int_0^\infty e^{-st} y(t) dt = \int_0^\infty e^{t^2-st} dt = \infty.$$

The exact real transform domain is empty.

Step 2: Solve the formal transformed equation. Pretending the rules apply gives $sY - 1 + 2Y' = 0$, so

$$Y' + \frac{s}{2}Y = \frac{1}{2}, \quad Y_C(s) = e^{-s^2/4} \left[C + \frac{1}{2} \int_0^s e^{u^2/4} du \right].$$

This is a valid family of solutions to the differential equation in s . It is not yet a family of Laplace transforms.

Step 3: Check the deceptively correct limits. As $s \rightarrow \infty$, l'Hopital's rule gives

$$\frac{\int_0^s e^{u^2/4} du}{2e^{s^2/4}/s} \rightarrow 1,$$

because the derivative ratio is $1/(1 - 2/s^2) \rightarrow 1$. Hence the integral contribution to Y_C is asymptotic to $1/s$, while $Ce^{-s^2/4}$ decays faster. Every real C therefore satisfies

$$Y_C(s) \rightarrow 0, \quad sY_C(s) \rightarrow 1.$$

These two necessary-looking conditions do not even select a unique formal branch here.

Step 4: Identify the invalid step. The ordinary integrals defining Y , the transform of ty , and the derivative transform do not converge for any real s . The time solution is not of exponential order, and direct growth proves failure, not merely absence of a sufficient hypothesis. Applying transform identities to these divergent integrals was unjustified. The limits in Step3 cannot repair that step. In contrast, changing the sign to $y' + 2ty = 0$ gives e^{-t^2} , whose transform exists for every real s and can be verified directly. A differential equation in transform space is useful only after its relationship to convergent time integrals is established.