

Question 2

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Solve

$$y' + 2ty = 0, \quad y(0) = 1, \quad t \geq 0.$$

You may leave Gaussian integrals unevaluated and use $\int_0^\infty e^{-v^2} dv = \sqrt{\pi}/2$.

Tasks

1. Derive the first-order equation for Y and solve it using an integrating factor.
2. Select the admissible transform branch by its behavior as $s \rightarrow \infty$.
3. Identify the time solution, verify the IVP, and prove that the Gaussian-integral expression is its forward transform for every real s .
4. Evaluate $Y(0)$ and prove $1/s - 2/s^3 \leq Y(s) \leq 1/s$ for $s > 0$. Interpret the resulting initial-value limit.

Question 2 – Solution

Strategy. The variable damping gives a growing homogeneous solution in transform space; a genuine transform selects the complementary integral tail.

Step 1: Transform and apply the integrating factor. The equation becomes $sY - 1 - 2Y' = 0$, or

$$Y' - \frac{s}{2}Y = -\frac{1}{2}, \quad (e^{-s^2/4}Y)' = -\frac{1}{2}e^{-s^2/4}.$$

Step 2: Impose the transform condition at infinity. For a bounded candidate, $Y = O(1/s)$, so $e^{-s^2/4}Y \rightarrow 0$. Integrating to infinity yields

$$Y(s) = \frac{1}{2}e^{s^2/4} \int_s^\infty e^{-u^2/4} du = e^{s^2/4} \int_{s/2}^\infty e^{-v^2} dv.$$

Any additional $Ce^{s^2/4}$ violates the bound. The recovered inverse will verify boundedness and the entire transform calculation.

Step 3: Identify the inverse and its full domain. Separation in time gives $y = e^{-t^2}$, with $y' = -2te^{-t^2}$ and $y(0) = 1$. Completing the square directly gives

$$\int_0^\infty e^{-st-t^2} dt = e^{s^2/4} \int_{s/2}^\infty e^{-v^2} dv.$$

The Gaussian tail is integrable for every real s , so this is the exact real domain. This direct verification also justifies the selected transformed branch. Uniqueness follows from the regular first-order equation.

Step 4: Check the area and large-parameter bound. At zero, $Y(0) = \sqrt{\pi}/2$. For $t \geq 0$, the elementary inequality $e^{-x} \geq 1 - x$ gives $1 - t^2 \leq e^{-t^2} \leq 1$. Integrating with the positive kernel yields

$$\frac{1}{s} - \frac{2}{s^3} \leq Y(s) \leq \frac{1}{s}, \quad s > 0.$$

The lower bound may be negative for small s but remains valid. Multiplying by s and taking $s \rightarrow \infty$ gives $sY(s) \rightarrow 1 = y(0)$. The graph shows the actual rapidly decaying time solution.

