

Question 1

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Consider the regular IVP

$$(1+t)y' + y = 1, \quad y(0) = 0, \quad t \geq 0.$$

An answer in terms of a convergent parameter integral is acceptable for Y .

Tasks

1. Derive the differential equation satisfied by Y . Differentiate the entire transform of y' when handling ty' .
2. Solve that equation using an integrating factor and a condition at $s = \infty$ that a bounded time function must satisfy.
3. Identify the inverse and verify the original IVP. Check your parameter-integral expression directly against its forward transform.
4. State the exact real transform domain and derive the leading large- s behavior from the initial slope.

Question 1 – Solution

Strategy. Multiplication by time makes a differential equation in the transform variable. Its integration constant must be selected using transform behavior.

Step 1: Differentiate the full derivative transform. Here $\mathcal{L}\{y'\} = sY$, so $\mathcal{L}\{ty'\} = -(Y + sY')$. The transformed equation is

$$sY - Y - sY' + Y = 1/s, \quad \boxed{Y' - Y = -1/s^2}, \quad s > 0.$$

Treating $\mathcal{L}\{ty'\}$ as only $-sY'$ would introduce an erroneous extra term.

Step 2: Select the integration constant. Multiply by e^{-s} . A bounded function has $Y = O(1/s)$ as $s \rightarrow \infty$, hence $e^{-s}Y \rightarrow 0$. Integration from s to infinity gives

$$\boxed{Y(s) = e^s \int_s^\infty \frac{e^{-u}}{u^2} du.}$$

The other homogeneous term is Ce^s , incompatible with that bound unless $C = 0$. The inverse identified below is bounded, so this selection is self-consistent and verified rather than assumed without a check.

Step 3: Identify and forward-check the inverse. The original left side is $[(1+t)y]'$, giving $(1+t)y = t$ and

$$\boxed{y(t) = \frac{t}{1+t}.}$$

Its derivative is $(1+t)^{-2}$, so $(1+t)y' + y = 1$ and $y(0) = 0$. For $s > 0$, substitution $u = s(1+t)$ gives $\mathcal{L}\{(1+t)^{-1}\} = e^s \int_s^\infty e^{-u}/u du$. Thus the forward transform is $1/s - e^s \int_s^\infty e^{-u}/u du$, equal to the boxed parameter integral by integration by parts. The regular first-order IVP is unique.

Step 4: Check domain and onset. Since $y \rightarrow 1$ and is nonnegative, its exact domain is $\boxed{s > 0}$. At the origin $y'(0) = 1$, and the substitution $v = st$ gives

$$s^2 Y(s) = \int_0^\infty e^{-v} \frac{v}{1+v/s} dv \rightarrow 1.$$

The integrand is dominated by ve^{-v} , so dominated convergence justifies the limit. Thus $Y \sim 1/s^2$, matching the zero initial value and unit initial slope.

