

Question 10

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

An input $q(t)$ is to produce a prescribed smooth rise in

$$y'' + 3y' + 2y = q(t), \quad y(0) = 0, \quad y'(0) = 0.$$

The target family is $y_k(t) = 1 - (1 + kt)e^{-kt}$, with $k > 0$. Allowed inputs have the form $q(t) = 2 + Ce^{-kt}$, with no term involving te^{-kt} .

Tasks

1. Find the target transform $Y_k(s)$ and derive the input transform required by the IVP.
2. Invert that input transform and determine exactly which $k > 0$ satisfy the allowed input form.
3. An additional requirement is $y(0) = 4$. Select k, C and verify the complete IVP with the resulting input.
4. Prove that the selected response increases to one without overshoot. Explain why the required input is unique once the full target trajectory is fixed.

Question 10 – Solution

Strategy. Transform the target first, then use the differential operator to recover the necessary input. Check the permitted input family before selecting parameters.

Step 1: Transform the prescribed trajectory. The exponential and time-exponential pairs give

$$Y_k = \frac{1}{s} - \frac{1}{s+k} - \frac{k}{(s+k)^2} = \boxed{\frac{k^2}{s(s+k)^2}}.$$

All targets have $y_k(0) = y'_k(0) = 0$, so their required input transform is

$$Q_k = (s^2 + 3s + 2)Y_k = \frac{k^2(s+1)(s+2)}{s(s+k)^2}.$$

Step 2: Recover the input and impose the restriction. Partial fractions, verified by clearing denominators, give

$$Q_k = \frac{2}{s} + \frac{k^2 - 2}{s+k} - \frac{k(k-1)(k-2)}{(s+k)^2},$$

$$q_k(t) = 2 + (k^2 - 2)e^{-kt} - k(k-1)(k-2)te^{-kt}.$$

For $k > 0$, the last term vanishes exactly for $\boxed{k=1 \text{ or } k=2}$. Their respective values of C are -1 and 2 . A nonzero te^{-kt} coefficient cannot be absorbed into a constant multiple of e^{-kt} on an interval.

Step 3: Select and verify the desired acceleration. Differentiation gives $y'_k = k^2te^{-kt}$ and $y''_k = k^2(1 - kt)e^{-kt}$, so $y''_k(0) = k^2$. The added requirement selects

$$\boxed{k=2, \quad C=2, \quad q=2+2e^{-2t}, \quad y=1-(1+2t)e^{-2t}}.$$

Here $y' = 4te^{-2t}$, $y'' = (4 - 8t)e^{-2t}$, and $y'' + 3y' + 2y = 2 + 2e^{-2t}$. The initial values are $0, 0$ and the acceleration is four, as required. Both Y and Q have exact domain $s > 0$ because their original functions approach positive constants.

Step 4: Prove the rise and uniqueness of the input. Since $y'(t) > 0$ for $t > 0$, the response increases from zero. Also $1 - y = (1 + 2t)e^{-2t} > 0$ for every finite t , and this difference tends to zero. Thus $\boxed{0 < y(t) < 1 \ (t > 0), \quad y(t) \rightarrow 1}$ with no overshoot. Once the entire twice-differentiable target is fixed, the equation forces $q = y'' + 3y' + 2y$ pointwise. The input is unique, rather than merely one successful choice; linear-IVP uniqueness then verifies that it produces this trajectory.

