

Question 9

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

Use Laplace transforms for the third-order IVP

$$y^{(3)} + 3y'' + 3y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0.$$

Tasks

1. Derive the transformed equation, displaying the contributions of all three initial data.
2. Invert the result at the repeated pole, and check all three data.
3. Verify the third-order differential equation directly, using $y = e^{-t}v$ if useful.
4. Prove positivity and strict decrease for $t > 0$. Compute the total integral and exact transform domain, and explain why the initial flatness does not make the solution constant.

Question 9 – Solution

Strategy. Higher derivatives introduce more initial terms. A repeated transform pole packages the polynomial-exponential solution compactly.

Step 1: Transform the derivative hierarchy. The general third-derivative rule is $\mathcal{L}\{y^{(3)}\} = s^3Y - s^2y(0) - sy'(0) - y''(0)$. Here the first three derivatives transform to $sY - 1$, $s^2Y - s$ and $s^3Y - s^2$. Hence

$$(s^3Y - s^2) + 3(s^2Y - s) + 3(sY - 1) + Y = 0,$$

$$Y = \frac{s^2 + 3s + 3}{(s + 1)^3}.$$

Even the zero initial values must be placed in the correct derivative formulas before simplifying.

Step 2: Expand at the repeated pole. Writing $p = s + 1$ makes the numerator $p^2 + p + 1$, so

$$Y = \frac{1}{s + 1} + \frac{1}{(s + 1)^2} + \frac{1}{(s + 1)^3}, \quad \boxed{y = e^{-t}(1 + t + t^2/2)}.$$

We have $y' = -t^2e^{-t}/2$ and $y'' = e^{-t}(t^2/2 - t)$. Thus the three initial values are exactly 1, 0, 0.

Step 3: Verify the operator identity. For $y = e^{-t}v$, repeated product rules give

$$y^{(3)} + 3y'' + 3y' + y = e^{-t}v^{(3)}.$$

Here $v = 1 + t + t^2/2$ has third derivative zero. This verifies the equation everywhere. The linear third-order IVP with all three data has a unique solution.

Step 4: Check shape, area and domain. The polynomial factor is positive on $t \geq 0$, and $y' < 0$ for every $t > 0$, so y is positive and strictly decreasing after its initially flat point. It tends to zero. Direct exponential moments give

$$\boxed{\int_0^\infty y(t) dt = 1 + 1 + 1 = 3 = Y(0)}.$$

The exact transform domain is $s > -1$; at or below -1 the positive polynomial tail makes the integral diverge. The data $y'(0) = y''(0) = 0$ describe only the initial point. The equation gives $y^{(3)}(0) = -1$, so the first change is cubic rather than absent.