

Question 8

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

An accumulated quantity enters an IVP:

$$y'(t) + y(t) + \int_0^t y(u) du = 1, \quad y(0) = 0.$$

You may use $\mathcal{L}\{\int_0^t y(u) du\} = Y(s)/s$ where the integrals converge.

Tasks

1. Solve directly by Laplace transforms, including the integral term.
2. Find an explicit expression for $I(t) = \int_0^t y(u) du$ and verify the original equation.
3. Differentiate the original equation and determine the extra initial condition needed for the resulting second-order equation. Explain why omitting that condition enlarges the solution set.
4. Find the first positive zero of y , its limit, the limit of I , and the exact transform domain of Y . Explain why a constant right-hand side does not force a nonzero limiting y here.

Question 8 – Solution

Strategy. Transform the accumulated state as well as the instantaneous state. Verify the original integral equation after any differentiation.

Step 1: Transform and complete the square. For sufficiently large s , the initial value gives

$$sY + Y + Y/s = 1/s, \quad Y = \frac{1}{s^2 + s + 1}.$$

Let $b = \sqrt{3}/2$. Completing the square gives

$$y(t) = \frac{2}{\sqrt{3}} e^{-t/2} \sin(bt).$$

This inverts $Y = 1/[(s + 1/2)^2 + b^2]$ with the required sine normalization.

Step 2: Reconstruct the integral and verify. Integration, or differentiation of the following expression, gives

$$I(t) = 1 - e^{-t/2} [\cos(bt) + \frac{1}{\sqrt{3}} \sin(bt)].$$

It satisfies $I(0) = 0$ and $I' = y$. Also $y' = e^{-t/2} [\cos(bt) - \sin(bt)/\sqrt{3}]$. Substitution yields $y' + y + I = 1$ identically, and $y(0) = 0$. This checks the undifferentiated problem.

Step 3: Retain the compatibility condition. Differentiation gives $y'' + y' + y = 0$. Evaluating the original equation at zero gives $y'(0) = 1$. Thus the equivalent second-order IVP has data $y(0) = 0, y'(0) = 1$. If only $y(0) = 0$ is retained, a free initial slope remains; it sets the constant value of $y' + y + I$, which need not be one. Conversely, a solution of the differentiated equation with both correct data makes that quantity constant and equal to one, proving equivalence and uniqueness.

Step 4: Interpret the limiting balance. The first positive zero is $t = \pi/b = 2\pi/\sqrt{3}$. The function changes sign there and tends to zero, while $I(t) \rightarrow 1$. The accumulated state supplies the limiting balance, rather than a nonzero instantaneous state. The exact domain of Y is $s > -1/2$, from the damped sine and its nonconvergent boundary integral. The integral state's transform needs $s > 0$ because $I \rightarrow 1$; the initial derivation used this common domain. The wider domain of Y follows from the recovered function itself.

