

**Question 7**

Use one-sided Laplace transforms and retain all initial-value terms. Write  $Y(s) = \mathcal{L}\{y\}(s)$ , with real  $s$  sufficiently large during the transformation. Unless stated otherwise, solve on  $t \geq 0$ .

The data are prescribed at time one, not at zero:

$$y'' + y = t, \quad y(1) = 2, \quad y'(1) = 0, \quad t \geq 1.$$

Use a new time origin before applying a one-sided transform.

**Tasks**

1. Define  $\tau = t - 1$  and  $z(\tau) = y(1 + \tau)$ . Derive the transformed IVP for  $z$ .
2. Find  $Z(s) = \mathcal{L}_\tau\{z\}(s)$  and invert it.
3. Convert back to  $t$  and verify the equation and the data at time one.
4. A student treats the supplied values as  $y(0), y'(0)$  and transforms the forcing as  $1/s^2$ . Explain both changes needed to repair that calculation. State the exact domain of  $Z$ .

## Question 7 – Solution

**Strategy.** Shift the independent variable and the forcing together. A Laplace transform beginning at the new origin must use data at that origin.

**Step 1: Move the initial time.** With  $\tau = t - 1$ , the derivatives have the same form but the forcing becomes  $\tau + 1$ . Thus

$$z'' + z = \tau + 1, \quad z(0) = 2, \quad z'(0) = 0.$$

Writing  $Z = \mathcal{L}_\tau\{z\}$  gives

$$(s^2 + 1)Z - 2s = \frac{1}{s^2} + \frac{1}{s}.$$

**Step 2: Resolve and invert.** A useful decomposition is

$$Z = \frac{2s + 1/s^2 + 1/s}{s^2 + 1} = \frac{1}{s^2} + \frac{1}{s} + \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1}.$$

Therefore  $\boxed{z(\tau) = \tau + 1 + \cos \tau - \sin \tau}$ . Transforming each term back checks the decomposition, including the constant part of the shifted forcing.

**Step 3: Return to the original clock.** The solution on the requested domain is

$$\boxed{y(t) = t + \cos(t - 1) - \sin(t - 1), \quad t \geq 1.}$$

Its first derivative is  $1 - \sin(t - 1) - \cos(t - 1)$  and its second is  $-\cos(t - 1) + \sin(t - 1)$ . Hence  $y'' + y = t$ , while  $y(1) = 2$  and  $y'(1) = 0$ . These are the actual specified data, so uniqueness on  $[1, \infty)$  verifies the result.

**Step 4: Diagnose the two clock errors.** The values two and zero belong to  $z(0), z'(0)$ , not to values of  $y$  at an unspecified earlier time. Once that clock change is made, the forcing is  $\tau + 1$  and its transform is  $1/s^2 + 1/s$ , not just  $1/s^2$ . The transform being inverted is  $Z$ , defined using  $\tau \geq 0$ . Since  $z(\tau) = \tau + O(1)$  is eventually positive, it has exact domain  $\boxed{s > 0}$ . No extension of  $y$  to times before one was needed.