

Question 5

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

An unstable equation has adjustable initial velocity:

$$y'' - y = e^{-2t}, \quad y(0) = 1, \quad y'(0) = v.$$

Tasks

1. Derive and invert Y for arbitrary real v .
2. Find the unique v that gives a bounded solution, and verify the resulting IVP.
3. If the chosen velocity is perturbed by δ , derive the exact error in the solution. Determine the largest allowed $|\delta|$ that guarantees an error at most 0.01 for every $0 \leq t \leq 5$.
4. State the exact transform domains for the bounded choice and for every other v . Explain why the uncanceled operator factor alone does not determine the response domain.

Question 5 – Solution

Strategy. Initial data control the coefficient of the unstable mode. Exact cancellation can give a bounded trajectory but does not remove sensitivity to an initial error.

Step 1: Transform and resolve the modes. The transformed equation is

$$(s^2 - 1)Y = s + v + \frac{1}{s + 2}.$$

Partial fractions give

$$Y = \frac{v/2 + 2/3}{s - 1} + \frac{-v/2}{s + 1} + \frac{1/3}{s + 2},$$

$$y(t) = (v/2 + 2/3)e^t - (v/2)e^{-t} + \frac{1}{3}e^{-2t}.$$

Recombining the fractions reproduces the transformed equation.

Step 2: Cancel the growing coefficient. Boundedness requires $v/2 + 2/3 = 0$, so the unique choice is

$$v = -4/3, \quad y_b(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{-2t}.$$

Then $y_b(0) = 1$, $y'_b(0) = -2/3 - 2/3 = -4/3$, and $y''_b - y_b = e^{-2t}$. If the growing coefficient is nonzero, it eventually dominates both decaying terms, so no other velocity works.

Step 3: Quantify the initial-error tolerance. Replacing v by $-4/3 + \delta$ changes the solution by

$$y - y_b = \frac{\delta}{2}(e^t - e^{-t}) = \delta \sinh t.$$

Since $\sinh t$ increases on $t \geq 0$, the maximum absolute error on $[0, 5]$ is $|\delta| \sinh 5$. The exact, necessary and sufficient tolerance is

$$|\delta| \leq \frac{0.01}{\sinh 5} \approx 1.3477 \times 10^{-4}.$$

Equality is permitted and attains the error limit at time five. The graph compares the bounded choice with a small positive velocity error.

Step 4: Check the actual tails. For the bounded choice, the nonzero slow tail is $(2/3)e^{-t}$, so the exact domain is $s > -1$. For every other v , a nonzero e^t term gives exact domain $s > 1$. The operator factor $s - 1$ cancels from the selected response numerator. Its presence in the operator does not mean it survives in every solution; the initial data determine the residue.

