

Question 4

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

An undamped oscillator starts at rest:

$$y'' + 4y = \sin 2t, \quad y(0) = 0, \quad y'(0) = 0.$$

Define its energy by $E(t) = [y'(t)^2 + 4y(t)^2]/2$.

Tasks

1. Derive Y and invert it. You may use $\mathcal{L}\{t \cos 2t\} = (s^2 - 4)/(s^2 + 4)^2$.
2. Check the equation and both initial data directly.
3. Prove that the response is unbounded by evaluating a suitable explicit time sequence. State its exact real transform domain.
4. Derive the energy identity and calculate $E(n\pi)$ for positive integers n . Explain why bounded forcing can supply unbounded accumulated energy.

Question 4 – Solution

Strategy. Matching forcing and natural frequencies produce repeated transform poles. Explicit values and the work identity distinguish real growth from a mere upper bound.

Step 1: Transform and invert. Zero initial data give

$$(s^2 + 4)Y = \frac{2}{s^2 + 4}, \quad Y = \frac{2}{(s^2 + 4)^2}.$$

Combining the sine pair with the supplied time-cosine pair yields

$$y(t) = \frac{\sin 2t - 2t \cos 2t}{8}.$$

Indeed the forward numerator is $(s^2 + 4)/4 - (s^2 - 4)/4 = 2$.

Step 2: Verify the IVP. Direct differentiation gives

$$y' = \frac{t}{2} \sin 2t, \quad y'' = \frac{1}{2} \sin 2t + t \cos 2t.$$

Adding $4y = \frac{1}{2} \sin 2t - t \cos 2t$ gives $\sin 2t$. Both initial values are zero, proving the claimed solution by uniqueness.

Step 3: Exhibit unbounded values and the domain. At $t = n\pi$,

$$y(n\pi) = -n\pi/4 \longrightarrow -\infty.$$

The actual response is therefore unbounded; a growing envelope alone would not prove that. Its linear envelope gives absolute transform convergence for $s > 0$. At zero, its finite integral is $[1 - \cos 2R - R \sin 2R]/8$, which has no limit. Below zero, intervals around successive peaks of the $t \cos 2t$ term violate the Cauchy criterion. Hence the exact domain is $s > 0$.

Step 4: Account for the energy input. Multiplying the equation by y' gives

$$E' = y'(y'' + 4y) = y' \sin 2t = \frac{t}{2} \sin^2 2t \geq 0.$$

At $n\pi$, the velocity is zero, so

$$E(n\pi) = 2y(n\pi)^2 = n^2 \pi^2 / 8.$$

Although the forcing amplitude is at most one, its work accumulates over an increasing time interval. Resonance makes that accumulated work unbounded. The graph shows the growing displacement oscillations, not the energy itself.

