

Question 3

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

Consider a repeated-root operator with matching exponential forcing:

$$y'' + 2y' + y = e^{-t}, \quad y(0) = 2, \quad y'(0) = -2.$$

Tasks

1. Find Y and identify the order of the pole produced by the forcing.
2. Invert Y , keeping the factorial normalization of the repeated pole.
3. Verify the equation and both initial data directly. You may simplify the residual by setting $y = e^{-t}v$.
4. Determine whether y decreases to zero and whether $e^t y$ stays bounded. Explain why these conclusions are compatible and state the transform domain.

Question 3 – Solution

Strategy. A repeated operator root and a matching input pole combine. Pole order controls the polynomial factor, not necessarily growth of the full time response.

Step 1: Form the transformed equation. The initial terms give

$$(s^2Y - 2s + 2) + 2(sY - 2) + Y = \frac{1}{s + 1},$$

so

$$(s + 1)^2Y = 2(s + 1) + \frac{1}{s + 1}, \quad Y = \frac{2}{s + 1} + \frac{1}{(s + 1)^3}.$$

The forcing contributes a third-order pole at $s = -1$, although the differential equation has order two.

Step 2: Invert the repeated pole correctly. Since $\mathcal{L}\{t^2e^{-t}\} = 2/(s + 1)^3$,

$$y(t) = e^{-t}(2 + t^2/2).$$

The factor $1/2$ is essential. Transforming the two terms back gives exactly the displayed Y .

Step 3: Check the residual and data. With $v = 2 + t^2/2$, the product rule gives $y' = e^{-t}(v' - v)$ and $y'' = e^{-t}(v'' - 2v' + v)$. Therefore

$$y'' + 2y' + y = e^{-t}v'' = e^{-t}.$$

At zero, $v(0) = 2$, $v'(0) = 0$, so $y(0) = 2$ and $y'(0) = -2$. These checks and linear-IVP uniqueness verify the solution.

Step 4: Distinguish a scaled response from the actual response. We have

$$y' = e^{-t}(-2 + t - t^2/2) = -\frac{1}{2}e^{-t}[(t - 1)^2 + 3] < 0.$$

Thus y decreases strictly and tends to zero, since exponential decay dominates its quadratic factor. Meanwhile $e^ty = 2 + t^2/2 \rightarrow \infty$. Dividing out the decaying envelope changes the quantity being studied; there is no contradiction. The positive polynomial-exponential tail gives the exact domain $s > -1$, with divergence at and below the boundary.