

Question 1

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

Consider the IVP

$$y' + 2y = 3e^{-t}, \quad y(0) = 1.$$

A student writes $Y(s) = 3/[(s+1)(s+2)]$.

Tasks

1. Derive the correct transformed equation, and identify which IVP the student actually solved.
2. Find the correct inverse using partial fractions.
3. Check the original differential equation and initial value directly. State the exact real convergence interval of Y .
4. Find the unique maximum of y and its time. Explain why a decaying forcing does not force the response to decrease from the start.

Question 1 – Solution

Strategy. The derivative transform carries the initial value into the algebra. Keep that contribution before resolving the two poles.

Step 1: Include the initial term. Using $\mathcal{L}\{y'\} = sY - y(0)$ gives

$$(s+2)Y - 1 = \frac{3}{s+1}, \quad Y = \frac{1}{s+2} + \frac{3}{(s+1)(s+2)}.$$

The student omitted the first term. Their expression solves the same equation with initial value zero, not one.

Step 2: Resolve and invert. Combining the terms yields

$$Y = \frac{s+4}{(s+1)(s+2)} = \frac{3}{s+1} - \frac{2}{s+2}, \quad \boxed{y(t) = 3e^{-t} - 2e^{-2t}}.$$

Each reciprocal factor matches an elementary exponential transform.

Step 3: Verify the IVP and domain. We have $y(0) = 3 - 2 = 1$ and $y' = -3e^{-t} + 4e^{-2t}$. Consequently

$$y' + 2y = (-3 + 6)e^{-t} + (4 - 4)e^{-2t} = 3e^{-t}.$$

The coefficients and forcing are continuous, so the usual uniqueness theorem for this linear IVP makes the verified solution the unique one. Its tail is asymptotic to $3e^{-t}$ and is positive; hence its exact real transform domain is $\boxed{s > -1}$. The student instead obtains $3e^{-t} - 3e^{-2t}$, whose initial value is zero.

Step 4: Locate the overshoot. Since $y' = e^{-t}(-3 + 4e^{-t})$, the derivative is positive before and negative after $e^{-t} = 3/4$. Thus

$$\boxed{t_* = \ln(4/3), \quad y(t_*) = 9/8.}$$

Here $y'(0) = 1$ although the forcing decreases. The input initially exceeds the loss term $2y$, so the state first rises. It later decays to zero. The graph marks the overshoot above the initial level one.

