

Question 9

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

For an integer $n \geq 2$, define a right-continuous staircase

$$f_n(t) = \frac{k}{n} \quad \left(\frac{k}{n} \leq t < \frac{k+1}{n}, \quad k = 0, \dots, n-1\right), \quad f_n(t) = 0 \quad (t \geq 1).$$

The target is $r(t) = t$ on $0 \leq t < 1$ and $r(t) = 0$ for $t \geq 1$.

Tasks

1. Express f_n using steps and derive a finite-sum formula for its transform.
2. Find $F_n(0)$ and the target transform $R(s)$, including $R(0)$. Give the full real convergence sets.
3. For $s > 0$, derive a bound on $0 \leq R(s) - F_n(s)$ from the pointwise staircase error.
4. Using that bound, find the least integer n that certifies $R(s) - F_n(s) \leq 0.01$ simultaneously for every $s \geq 1$. Distinguish this certificate from an exact minimal-error calculation.

Question 9 – Solution

Strategy. Encode each upward increment and the final reset. A pointwise error bound becomes an integral certificate valid for a whole range of parameters.

Step 1: Include the final shutoff. There are increments $1/n$ at k/n , $1 \leq k \leq n-1$, followed by a drop $(n-1)/n$ at one. Hence

$$f_n(t) = \frac{1}{n} \sum_{k=1}^{n-1} u_{k/n}(t) - \frac{n-1}{n} u_1(t),$$

$$F_n(s) = \frac{\sum_{k=1}^{n-1} e^{-ks/n} - (n-1)e^{-s}}{ns} \quad (s \neq 0).$$

Transforming steps gives the formula for $s > 0$; finite-interval integration gives it for all $s \neq 0$.

Step 2: Evaluate area and the target. Each stair has width $1/n$, so

$$F_n(0) = \frac{1}{n^2} \sum_{k=0}^{n-1} k = \frac{n-1}{2n}.$$

Integration by parts on $[0, 1]$ gives

$$R(s) = \frac{1 - (1+s)e^{-s}}{s^2} \quad (s \neq 0), \quad R(0) = \frac{1}{2}.$$

Both original functions have finite support, so both transform domains are the whole real line. Their apparent singularities at zero are removable.

Step 3: Convert pointwise error to weighted error. On each half-open stair, $0 \leq r - f_n < 1/n$, and both functions vanish from one onward. For $s > 0$,

$$0 \leq R(s) - F_n(s) \leq \frac{1}{n} \int_0^1 e^{-st} dt = \frac{1 - e^{-s}}{ns}.$$

At zero the actual error is $1/(2n)$, while the continuous limiting bound is $1/n$. The bound is sufficient, not an equality.

Step 4: Certify a simultaneous tolerance. The function $b(s) = \int_0^1 e^{-st} dt$ is decreasing for $s > 0$, since its derivative is $-\int_0^1 te^{-st} dt < 0$. Its largest value on $s \geq 1$ is $1 - e^{-1}$. Thus the certificate requires

$$n \geq 100(1 - e^{-1}) \approx 63.2121, \quad \boxed{n = 64}.$$

For $n = 63$ this bound exceeds .01 at $s = 1$; that does not prove the actual error exceeds .01. The graph shows $n = 4$ only to display the staircase geometry; it is not the certified design.

