

Question 8

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

A three-stage pulse is one on $[0, 1)$, has unknown height B on $[1, 2)$ and unknown height C on $[2, 3)$, and is zero for $t \geq 3$. It must satisfy

$$\int_0^\infty f(t) dt = 0, \quad \int_0^\infty t f(t) dt = 0.$$

Tasks

1. Determine B, C and prove that these heights are uniquely determined.
2. Write the resulting pulse using steps and derive its transform in factored form.
3. Find the full real convergence set, the first nonzero term of $F(s)$ near zero, and the second time moment.
4. Explain why the two vanishing moments do not make the pulse or its transform identically zero. Determine the sign of $F(s)$ for $s > 0$.

Question 8 – Solution

Strategy. Moment constraints form a small linear system. The step differences then expose cancellation in the transform numerator.

Step 1: Solve the two moment equations. The unit-width intervals give

$$1 + B + C = 0, \quad \frac{1}{2} + \frac{3}{2}B + \frac{5}{2}C = 0.$$

Subtracting the first equation from twice the second yields $2B + 4C = 0$. Combining this with $B + C = -1$ gives $\boxed{B = -2, C = 1}$. The coefficient determinant is $5/2 - 3/2 = 1 \neq 0$, so the solution is unique.

Step 2: Encode changes and factor the transform. The levels 1, -2, 1, 0 have jumps $-3, +3, -1$. Hence

$$f(t) = 1 - 3u_1(t) + 3u_2(t) - u_3(t),$$

$$\boxed{F(s) = \frac{1 - 3e^{-s} + 3e^{-2s} - e^{-3s}}{s} = \frac{(1 - e^{-s})^3}{s} \quad (s \neq 0).}$$

This follows first by transforming steps for $s > 0$, or directly on the three finite intervals for any $s \neq 0$.

Step 3: Interpret the removable zero. Compact support gives convergence for every real s . Since $1 - e^{-s} = s + O(s^2)$,

$$\boxed{F(s) = s^2 + O(s^3), \quad F(0) = F'(0) = 0, \quad F''(0) = 2.}$$

Directly, the second moment is

$$\int_0^3 t^2 f(t) dt = \frac{1}{3} - 2\left(\frac{7}{3}\right) + \frac{19}{3} = \boxed{2},$$

consistent with differentiating the finite-support transform twice.

Step 4: Separate two constraints from identity. The pulse is one throughout its first interval, so it is not zero. For $s > 0$, both $1 - e^{-s}$ and s are positive, giving $\boxed{F(s) > 0}$. The zero area and first moment eliminate only the constant and linear terms near $s = 0$. They do not eliminate all weighted integrals. The second moment already witnesses the remaining information. The graph makes the signed cancellation visible without replacing it by a pointwise sign claim.

