

Question 6

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

A function has a deliberately exceptional point value:

$$f(t) = \begin{cases} 0, & 0 \leq t < 1, \\ 7, & t = 1, \\ e^{-(t-1)}, & t > 1. \end{cases}$$

Compare it with $g(t) = u_1(t)e^{-(t-1)}$.

Tasks

1. Compare f and g pointwise, including their left and right limits at one.
2. Find their ordinary transforms and exact convergence intervals.
3. Give a pointwise formula for f using g and the indicator of the singleton $\{1\}$. Explain why a finite sum of right-continuous step terms times continuous functions cannot reproduce this exceptional point value.
4. If a step convention instead assigns value $1/2$ at the switch, what changes in the formula for g and what stays the same in its transform? State precisely the uniqueness conclusion justified here.

Question 6 – Solution

Strategy. Separate equality at every point from equality except at isolated points. An ordinary integral cannot recover the latter point values.

Step 1: Locate the discrepancy. For $t \neq 1$, $f(t) = g(t)$. At one, $g(1) = 1$ but $f(1) = 7$. Both functions have left limit zero and right limit one. Thus g is right-continuous, whereas the chosen value of f agrees with neither limit.

Step 2: Transform the interval contribution. A single changed value contributes nothing to an integral. Substituting $v = t - 1$ in the tail gives, for both functions,

$$F(s) = G(s) = \frac{e^{-s}}{s+1}, \quad s > -1.$$

At $s = -1$, the weighted tail is the positive constant e ; below -1 it grows exponentially. Thus the stated convergence interval is exact. The isolated value seven cannot change this conclusion.

Step 3: Give a pointwise correction and prove the limitation. If $\mathbf{1}_{\{1\}}$ is one only at $t = 1$ and zero elsewhere, then

$$f(t) = g(t) + 6\mathbf{1}_{\{1\}}(t).$$

Each u_a under our convention is right-continuous. Multiplying it by a continuous function preserves right-continuity, and finite sums preserve that property. Such a sum must equal its right limit at one, so it cannot have the specified value seven while its right limit is one. This is a representational obstruction, not a transform-calculation error.

Step 4: Change the convention without changing the integral. With switch value $1/2$, the corresponding expression for g equals $1/2$ at one and is unchanged elsewhere. Its ordinary transform is still $e^{-s}/(s+1)$. A pointwise correction to recover f would now be $(13/2)\mathbf{1}_{\{1\}}$. Thus transform data alone do not specify the value at an isolated discontinuity. Here the right-hand continuous representative is fixed by the chosen convention; an unrestricted pointwise uniqueness claim would be false. The usual uniqueness statement at continuity points is consistent with all these examples.