

Question 4

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

Let $r_a(t) = (t - a)u_a(t)$. A continuous triangular signal is

$$f(t) = \begin{cases} 0, & 0 \leq t < 1, \\ t - 1, & 1 \leq t < 3, \\ (7 - t)/2, & 3 \leq t < 7, \\ 0, & t \geq 7. \end{cases}$$

Tasks

1. Express f as a linear combination of r_1, r_3, r_7 by tracking slope changes.
2. Find $F(s)$ using shifted ramp transforms, and establish its full real convergence set.
3. Explain algebraically why the ramp combination is zero after time seven. State the two cancellation conditions for a general sum $\sum c_j r_{a_j}(t)$ to vanish after its last switch.
4. Compute the total area and the first moment. Confirm the values from the expansion of F at zero.

Question 4 – Solution

Strategy. A ramp changes slope rather than level. Cancel both the final slope and the final intercept to create compact support.

Step 1: Encode slope changes. The slopes are 0, 1, $-1/2$, 0, giving changes 1, $-3/2$, $1/2$. Thus

$$f = r_1 - \frac{3}{2}r_3 + \frac{1}{2}r_7.$$

Each ramp is zero at its own switch, so the combination is continuous. At the peak time three its value is two; it then decreases to zero at seven.

Step 2: Transform the ramps. For $s > 0$, substitution gives $\mathcal{L}\{r_a\} = e^{-as}/s^2$. Hence

$$F(s) = \frac{e^{-s} - \frac{3}{2}e^{-3s} + \frac{1}{2}e^{-7s}}{s^2} \quad (s \neq 0).$$

The actual function has finite support, so its integral exists for all real s . Direct integration of the two linear pieces extends the formula to all $s \neq 0$; zero is removable.

Step 3: Cancel the tail, not just the slope. For $t \geq 7$, the ramp sum is

$$(1 - \frac{3}{2} + \frac{1}{2})t - [1 - \frac{3}{2}(3) + \frac{1}{2}(7)] = 0.$$

In general, after the last switch a finite ramp sum equals $t \sum c_j - \sum c_j a_j$. It vanishes identically there exactly when

$$\sum c_j = 0, \quad \sum c_j a_j = 0.$$

Canceling only the slope could leave a nonzero constant tail and a different convergence domain.

Step 4: Check area and moment. The triangle has base six and height two, so $F(0) = 6$. Direct integration gives

$$\int_1^3 t(t-1) dt = \frac{14}{3}, \quad \frac{1}{2} \int_3^7 t(7-t) dt = \frac{52}{3}.$$

Thus the first moment is $\frac{22}{3}$ and $F'(0) = -22$. Expanding the numerator of F gives $6s^2 - 22s^3 + O(s^4)$, agreeing with both results. The horizontal time coordinate of the area centroid is $22/6 = 11/3$, consistent with the three triangle vertices.

