

Question 3

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

An oscillator is observed only during the window

$$f(t) = \sin t [u_{\pi/2}(t) - u_{3\pi/2}(t)].$$

Use $\mathcal{L}\{\cos t\} = s/(s^2 + 1)$ for $s > 0$.

Tasks

1. Write the exact piecewise function, including its values at both window endpoints.
2. Express each switched sine using its local clock and derive $F(s)$ for $s > 0$.
3. Determine the full real convergence set, $F(0)$ and $F'(0)$. Verify them by direct finite-interval integration.
4. Explain why $F(s) > 0$ for $s > 0$ even though the signal has both signs and zero unweighted integral. Use the window symmetry to justify the sign.

Question 3 – Solution

Strategy. Re-express the phase at each switch separately. The second switch occurs at a different oscillator phase.

Step 1: Identify the window and endpoint values. The function equals $\sin t$ on $[\pi/2, 3\pi/2]$ and zero elsewhere. In particular $f(\pi/2) = 1$, while $f(3\pi/2) = 0$. At the latter time the left limit is -1 ; the graph distinguishes it from the assigned value.

Step 2: Shift the two phases correctly. With $a = \pi/2$ and $b = 3\pi/2$, we have $\sin(v + a) = \cos v$ and $\sin(v + b) = -\cos v$. Therefore

$$f = u_a(t) \cos(t - a) + u_b(t) \cos(t - b),$$

$$F(s) = \frac{s(e^{-\pi s/2} + e^{-3\pi s/2})}{s^2 + 1}.$$

The plus sign in the transform reflects the phase reversal at shutoff. After b , the two shifted cosine terms cancel exactly, as the original window requires.

Step 3: Use finite support to evaluate zero. The defining integral has finite support, so it converges for every real s ; finite-interval integration gives the displayed expression on the whole real line. Directly,

$$F(0) = \int_a^b \sin t \, dt = 0,$$

$$F'(0) = - \int_a^b t \sin t \, dt = -[-t \cos t + \sin t]_a^b = \boxed{2}.$$

The formula also gives $F(0) = 0$ and expands as $2s + O(s^2)$, verifying the derivative.

Step 4: Pair the equal and opposite lobes. Write $t = \pi + x$ with $-\pi/2 \leq x \leq \pi/2$. Pair x with $-x$. For $0 < x < \pi/2$, the earlier value is $\sin x > 0$ and the later value is $-\sin x$. Their weighted contribution is

$$e^{-\pi s} \sin x [e^{sx} - e^{-sx}] > 0 \quad (s > 0).$$

Integrating these pairs proves strict positivity, despite cancellation at $s = 0$. This also agrees with the positive numerator and denominator of the formula for $s > 0$.

