

Question 2

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

Compare the switched-on and delayed signals

$$p(t) = t^2 u_2(t), \quad q(t) = (t - 2)^2 u_2(t).$$

You may use $\mathcal{L}\{1\} = 1/s$, $\mathcal{L}\{t\} = 1/s^2$ and $\mathcal{L}\{t^2\} = 2/s^3$ for $s > 0$.

Tasks

1. Derive the shift formula for $u_a(t)g(t - a)$ by substitution in the defining integral.
2. Find the transforms of p and q . Rewrite the switched-on polynomial in its local time before using the formula.
3. A student assigns $2e^{-2s}/s^3$ to both functions. Identify the missing terms and check the values immediately after the switch.
4. Give the exact real convergence domains. Determine the leading large- s behavior of $e^{2s}P(s)$ and $e^{2s}Q(s)$ and relate it to the local onset.

Question 2 – Solution

Strategy. The factor e^{-as} shifts the time origin of the entire accompanying inverse, not just its domain of activity.

Step 1: Derive the delayed-shape rule. For parameters where the integral converges, substitute $v = t - a$:

$$\int_a^\infty e^{-st} g(t-a) dt = e^{-as} \int_0^\infty e^{-sv} g(v) dv.$$

Hence $\boxed{\mathcal{L}\{u_a(t)g(t-a)\} = e^{-as}G(s)}$. The effective shape is a function of elapsed time since the switch.

Step 2: Rewrite the unshifted polynomial. For $t \geq 2$, write $t = v + 2$. Then p has local shape $(v+2)^2 = v^2 + 4v + 4$, whereas q has local shape v^2 . Therefore

$$\boxed{P(s) = e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right), \quad Q(s) = \frac{2e^{-2s}}{s^3}, \quad s > 0.}$$

Direct substitution in each defining integral verifies these expressions.

Step 3: Identify the omitted local terms. The difference is

$$p - q = [4(t-2) + 4]u_2(t), \quad P - Q = e^{-2s}(4/s^2 + 4/s).$$

At the switch, $p(2) = 4$ while $q(2) = 0$, equal to their respective right limits. The switched-on signal jumps from zero to four; the delayed square starts continuously at zero with zero right slope. The student's expression describes only the latter.

Step 4: Check domains and onset scales. Both signals have nonnegative quadratic tails, so they converge absolutely for $s > 0$ and diverge for $s \leq 0$. Their exact domain is $\boxed{s > 0}$. Removing the delay factor gives

$$e^{2s}P(s) \sim 4/s, \quad e^{2s}Q(s) = 2/s^3 \quad (s \rightarrow \infty).$$

The constant local onset of p produces the $1/s$ term; the quadratic local onset of q produces $2/s^3$. A common switch time does not imply a common transformed shape.