

Question 9

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Consider

$$G(s) = \frac{s^2 + 2s + 2}{(s + 1)^2}.$$

Work only with ordinary integrals and continuous functions of exponential order. You may use that such a function has a global bound $|g(t)| \leq Me^{ct}$ on $t \geq 0$ for some $M > 0$ and real c .

Tasks

1. Divide the polynomials to separate G into its polynomial part and a strictly proper rational part.
2. Prove from the stated bound that every allowed ordinary transform tends to zero as $s \rightarrow \infty$. Decide whether G has an inverse in the stated class.
3. Find the inverse h of the strictly proper remainder and verify it directly. Explain why reporting h as the inverse of the whole G would be incorrect.
4. Define $k(t) = h(t)$ except that $k(1) = 7$. Do k and h have the same ordinary transform? Explain why this does not contradict uniqueness in the stated class.

Question 9 – Solution

Strategy. Check whether an ordinary inverse can exist before matching a rational remainder to a table. Keep the continuity hypothesis in any uniqueness claim.

Step 1: Divide before attempting inversion. Because $(s+1)^2 = s^2 + 2s + 1$,

$$G(s) = 1 + \frac{1}{(s+1)^2}.$$

The nonzero constant part cannot be silently discarded. In particular $G(s) \rightarrow 1$, rather than zero, at large positive s .

Step 2: Apply a necessary existence test. For an allowed g and $s > c$, absolute convergence and the global bound yield

$$|\mathcal{L}\{g\}(s)| \leq \int_0^\infty e^{-st} |g(t)| dt \leq M \int_0^\infty e^{-(s-c)t} dt = \frac{M}{s-c} \rightarrow 0.$$

A continuous function of exponential order admits this global bound by enlarging M on an initial compact interval. Since G has limit one, G has no inverse in the stated class. This conclusion concerns ordinary continuous functions under the explicit hypotheses.

Step 3: Invert only the actual remainder. The strictly proper part has inverse $h(t) = te^{-t}$. Direct integration by parts gives

$$\int_0^\infty te^{-(s+1)t} dt = \frac{1}{(s+1)^2}, \quad s > -1.$$

The integral diverges at or below -1 . Its transform is $G - 1$, not G . The missing constant does not arise from changing $h(0)$: changing one finite point value changes no ordinary integral.

Step 4: State uniqueness with its hypothesis. Here $h(1) = e^{-1} \neq 7$. Changing that one value leaves every finite-interval integral, and hence every convergent improper transform, unchanged. Thus k and h have identical transforms on $s > -1$. However, k is discontinuous at one, so it is not in the continuous class specified in the question. There is no contradiction: uniqueness within that class compares two continuous functions. The example separates equality of transforms from unrestricted pointwise equality. No generalized transform is needed to reach any of these conclusions.