

Question 8

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

An inverse is to be selected from the rational family

$$F_a(s) = \frac{as + 1}{(s + 1)^2}, \quad a \in \mathbb{R}.$$

The design requires a nonnegative continuous function with total integral one. Among such functions, one may also require it to be nonincreasing.

Tasks

1. Find the inverse for all a and verify its total integral and initial value.
2. Determine the exact set of a for which the inverse is nonnegative on $[0, \infty)$. Prove necessity as well as sufficiency.
3. Within that set, classify precisely when it is nonincreasing. For the other admissible parameters, locate the unique maximum.
4. The additional measurement $\int_0^\infty t f_a(t) dt = 3/2$ is exact. Determine a and decide whether all design requirements hold.

Question 8 – Solution

Strategy. Invert first, then turn global sign and monotonicity requirements into inequalities for a linear time factor.

Step 1: Separate the numerator at its pole. Since $as + 1 = a(s + 1) + (1 - a)$,

$$f_a(t) = e^{-t}[a + (1 - a)t].$$

Forward transformation gives $a/(s + 1) + (1 - a)/(s + 1)^2 = F_a$. The exact domain is $s > -1$ for every a : the nonzero linear or constant polynomial tail has eventual fixed sign and diverges at or below -1 . Thus $s = 0$ is allowed, and

$$\int_0^\infty f_a(t) dt = F_a(0) = 1, \quad f_a(0) = a.$$

Unit signed integral alone does not establish nonnegativity.

Step 2: Classify nonnegative inverses. Because $e^{-t} > 0$, the sign is that of $a + (1 - a)t$. Nonnegativity at zero requires $a \geq 0$. If $a > 1$, the negative slope makes the factor negative for sufficiently large t . Conversely, $0 \leq a \leq 1$ makes both its intercept and slope nonnegative. Therefore the exact admissible set is $0 \leq a \leq 1$.

Step 3: Distinguish monotone and peaked designs. For admissible a ,

$$f'_a(t) = e^{-t}[1 - 2a - (1 - a)t].$$

The bracket is nonincreasing in t , so it is nonpositive everywhere exactly when its initial value is nonpositive. Hence the nonincreasing designs have $1/2 \leq a \leq 1$. For $0 \leq a < 1/2$, the derivative changes from positive to negative at

$$t_* = \frac{1 - 2a}{1 - a}, \quad f_a(t_*) = (1 - a) \exp\left(-\frac{1 - 2a}{1 - a}\right).$$

This also covers $a = 0$, whose maximum occurs at one.

Step 4: Use the extra integral to select a design. Direct exponential moments give

$$\int_0^\infty t f_a(t) dt = a + 2(1 - a) = 2 - a.$$

Equating this to $3/2$ yields $a = 1/2$ and $f(t) = (1 + t)e^{-t}/2$. This is nonnegative, has unit integral, and is nonincreasing: $f'(t) = -te^{-t}/2 \leq 0$. Its derivative vanishes only at the initial endpoint. The measurement selects one member, rather than allowing the initial value to be prescribed independently.