

Question 7

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

A parameter-dependent expression is

$$F_\lambda(s) = \frac{s - 2 + \lambda}{(s - 2)(s + 1)}, \quad \lambda \in \mathbb{R}.$$

When a factor cancels, interpret its algebraic value at the canceled point by continuous extension.

Tasks

1. Find the inverse for every real λ and verify its transform.
2. Determine exactly which parameter values give a bounded inverse. State the full real convergence domain in every case.
3. Explain why the printed factor $s - 2$ alone does not establish a growing mode when $\lambda = 0$. What transform value exists at $s = 2$ in that case?
4. For $\lambda = 0.03$, find the time at which the growing and decaying contributions are equal. Explain how a small coefficient can change the eventual behavior.

Question 7 – Solution

Strategy. Residues, including their possible zeros, decide which modes are actually present. An unreduced denominator can mislead.

Step 1: Recover both mode coefficients. Write $F_\lambda = A/(s-2) + B/(s+1)$. Clearing denominators gives $A + B = 1$ and $A - 2B = -2 + \lambda$, so

$$f_\lambda(t) = \frac{\lambda}{3}e^{2t} + (1 - \frac{\lambda}{3})e^{-t}.$$

Forward transformation recombines to $(A+B)s + (A-2B) = s - 2 + \lambda$, as required, for sufficiently large s .

Step 2: Classify boundedness and domains. If $\lambda = 0$, the function is e^{-t} , bounded and decaying, with exact transform domain $s > -1$. If $\lambda \neq 0$, then $e^{-2t}f_\lambda(t) \rightarrow \lambda/3 \neq 0$. Thus it is unbounded and its weighted tail at $s \leq 2$ has an eventual fixed sign and nonintegrable magnitude. For $s > 2$, all remaining terms converge absolutely. Consequently

bounded exactly when $\lambda = 0$;

$$\begin{array}{ll} s > -1, & \lambda = 0, \\ s > 2, & \lambda \neq 0. \end{array}$$

This includes $\lambda = 3$, when the decaying term disappears.

Step 3: Interpret the canceled point. At $\lambda = 0$, cancellation gives $F_0 = 1/(s+1)$. The apparent pole at two was never a pole of the reduced transform. The actual integral at $s = 2$ is

$$\int_0^\infty e^{-2t}e^{-t} dt = 1/3.$$

The continuous algebraic extension agrees with it. The inverse is determined by the whole expression, not by a list of uncanceled denominator factors.

Step 4: Measure the delayed dominance. For $\lambda = .03$, the inverse is $.01e^{2t} + .99e^{-t}$. Equality of the contributions requires $e^{3t} = 99$, so

$$t_c = \frac{1}{3} \ln 99 \approx 1.5317.$$

The ratio of growing to decaying contributions is $e^{3t}/99$, strictly increasing through one. The initially small growing coefficient eventually dominates. At fixed finite time the inverse approaches e^{-t} as $\lambda \rightarrow 0$, but for any nonzero λ its eventual boundedness is different. That distinction does not justify discarding a small nonzero residue.