

Question 6

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

For $a > 0$, let

$$F_a(s) = \frac{1}{(s+1)(s+a)}.$$

The two poles coincide when $a = 1$.

Tasks

1. Find the inverse for $a \neq 1$ and separately for $a = 1$. Explain why substitution into the unsimplified partial fractions fails at $a = 1$.
2. Prove the inverse is positive for every $t > 0$, and give its exact real convergence interval for all $a > 0$.
3. Derive the representation $f_a(t) = te^{-t} \int_0^1 e^{-(a-1)tv} dv$ directly from your formula, including the case $a = 1$.
4. Use the representation to prove uniform convergence $f_a \rightarrow f_1$ on the whole half-line as $a \rightarrow 1$. Obtain an explicit bound proportional to $|a - 1|$ for $1/2 \leq a \leq 3/2$.

Question 6 – Solution

Strategy. Combine the two large opposite coefficients before taking the coincident-pole limit. An integral representation controls the difference uniformly.

Step 1: Invert before and at collision. For $a \neq 1$, partial fractions give

$$F_a = \frac{1}{a-1} \left(\frac{1}{s+1} - \frac{1}{s+a} \right), \quad \boxed{f_a(t) = \frac{e^{-t} - e^{-at}}{a-1}}.$$

At $a = 1$, $F_1 = (s+1)^{-2}$, so $\boxed{f_1(t) = te^{-t}}$. Both forward transforms verify the original expression. The two individual partial-fraction coefficients diverge as $a \rightarrow 1$; their combined inverse has a finite limit.

Step 2: Check sign and the slowest tail. For $a > 1$, numerator and denominator of f_a are positive when $t > 0$; for $0 < a < 1$, both are negative. Also $te^{-t} > 0$. For unequal poles the slower exponential has a positive, nonzero coefficient, while the other term decays faster. Thus its tail is bounded below by a positive multiple of $e^{-\min(1,a)t}$. Together with direct absolute convergence, this proves the exact domain

$$\boxed{s > -\min(1, a)}.$$

At $a = 1$, the same statement follows from the positive polynomial-exponential tail.

Step 3: Remove the cancellation analytically. Integrating $e^{-(a-1)tv}$ in v gives

$$te^{-t} \int_0^1 e^{-(a-1)tv} dv = \frac{e^{-t} - e^{-at}}{a-1}$$

when $a \neq 1$ and $t > 0$. At $t = 0$ both sides extend to zero; at $a = 1$ the integral is one, yielding te^{-t} . This representation avoids subtracting nearly equal exponentials.

Step 4: Prove uniform half-line convergence. The mean value theorem gives $|e^{-x} - 1| \leq |x|e^{\max(0, -x)}$. With $m = \min(1, a)$, the representation yields

$$|f_a(t) - te^{-t}| \leq \frac{|a-1|}{2} t^2 e^{-mt}.$$

The maximum of $t^2 e^{-mt}$ on $t \geq 0$ occurs at $t = 2/m$ and equals $4/(m^2 e^2)$. Therefore, for $1/2 \leq a \leq 3/2$,

$$\boxed{\sup_{t \geq 0} |f_a(t) - f_1(t)| \leq \frac{8}{e^2} |a-1| \rightarrow 0}.$$

This is uniform on the entire half-line, stronger than checking each fixed time. The plotted nearby curves meet at the same zero initial value.

