

Question 5

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Consider the repeated quadratic factor

$$F(s) = \frac{1}{(s^2 + 4)^2}.$$

You may use $\mathcal{L}\{\sin 2t\} = 2/(s^2 + 4)$, $\mathcal{L}\{\cos 2t\} = s/(s^2 + 4)$ and $\mathcal{L}\{tg(t)\} = -G'(s)$ for $s > 0$.

Tasks

1. Derive the transform of $t \cos 2t$ by differentiating the cosine transform.
2. Find constants A, B such that the inverse is $A \sin 2t + Bt \cos 2t$. Verify the resulting rational numerator.
3. Show that the inverse is unbounded despite having no positive exponential factor. Give an explicit sequence witnessing growth.
4. Find its leading term near $t = 0$ and its exact real convergence interval. Compare the local term with the large- s scale $F(s) \sim s^{-4}$.

Question 5 – Solution

Strategy. A repeated quadratic can be handled with parameter differentiation and linear combinations, without assuming a product rule for inverses.

Step 1: Produce the repeated denominator. Differentiating the cosine pair gives

$$\mathcal{L}\{t \cos 2t\} = -\frac{d}{ds} \frac{s}{s^2 + 4} = \frac{s^2 - 4}{(s^2 + 4)^2}.$$

Polynomial factors times the damping kernel are integrable for $s > 0$, justifying the differentiation.

Step 2: Cancel the unwanted numerator. The transform of $A \sin 2t + Bt \cos 2t$ has numerator $2A(s^2 + 4) + B(s^2 - 4)$. Requiring it to equal one gives $2A + B = 0$, $8A - 4B = 1$, hence

$$f(t) = \frac{\sin 2t - 2t \cos 2t}{16}.$$

Substitution gives $(s^2 + 4)/8 - (s^2 - 4)/8 = 1$, verifying the numerator exactly. Inverting a product by multiplying inverses would not give this result.

Step 3: Exhibit growth, not just an envelope. The bound $|f(t)| \leq (1 + 2t)/16$ is only an upper bound. Actual unboundedness follows from

$$f(n\pi) = -n\pi/8 \longrightarrow -\infty.$$

The factor t produced by the repeated poles causes increasing oscillation size. The graph displays several oscillations at their actual scale.

Step 4: Check onset and convergence. Taylor expansion gives $\sin 2t - 2t \cos 2t = (8/3)t^3 + O(t^5)$, so $f(t) = t^3/6 + O(t^5)$. Since $\mathcal{L}\{t^3/6\} = s^{-4}$, the initial scale agrees with $F(s) \sim s^{-4}$. Absolute convergence holds for $s > 0$ by the linear envelope. At $s = 0$, direct integration gives

$$\int_0^R f(t) dt = \frac{1 - \cos 2R - R \sin 2R}{16},$$

which has no limit. For $s < 0$, intervals near successive positive peaks of $-\cos 2t$ have eventually positive integrand of order $te^{|s|t}$; their integrals fail the Cauchy criterion. Thus $\boxed{s > 0}$ is exact.

