

Question 4

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Let

$$F(s) = \frac{5}{(s+1)(s^2+4)}.$$

The required basic pairs are e^{-t} , $\cos 2t$ and $\sin 2t$.

Tasks

1. Choose a complete real partial-fraction form and solve for its coefficients.
2. Find the inverse and verify its forward transform by recombining numerators.
3. Compute the first three initial quantities $f(0), f'(0), f''(0)$. Relate their cancellations to the leading large- s behavior of F .
4. Determine the exact real convergence interval. Does the inverse approach a limit as $t \rightarrow \infty$? Justify your answer using explicit sequences.

Question 4 – Solution

Strategy. An irreducible quadratic requires a linear numerator. The transient and persistent parts then have different long-time roles.

Step 1: Resolve the real factors. Use

$$F = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}.$$

Clearing denominators gives $5 = A(s^2+4) + (Bs+C)(s+1)$, so $A+B=0$, $B+C=0$, $4A+C=5$. Hence $A=1$, $B=-1$, $C=1$.

Step 2: Invert each numerator term. Since the sine pair has numerator two,

$$f(t) = e^{-t} - \cos 2t + \frac{1}{2} \sin 2t.$$

The forward numerator is $(s^2+4) - s(s+1) + (s+1) = 5$, giving the required transform. A constant numerator over the quadratic alone would have omitted the cosine contribution.

Step 3: Check the onset cancellations. Direct differentiation gives

$$f' = -e^{-t} + 2 \sin 2t + \cos 2t, \quad f'' = e^{-t} + 4 \cos 2t - 2 \sin 2t.$$

Therefore $f(0) = 0$, $f'(0) = 0$, $f''(0) = 5$. Also $F(s) = 5/s^3 + O(s^{-4})$: the $1/s$ and $1/s^2$ terms vanish. Repeated integration by parts connects precisely these coefficients to the displayed initial values. In time, the first nonzero Taylor term is $5t^2/2$.

Step 4: Identify the persistent oscillation. The exponential term decays, but $-\cos 2t + \frac{1}{2} \sin 2t$ does not. In fact,

$$f(n\pi) \longrightarrow -1, \quad f(n\pi + \pi/2) \longrightarrow 1.$$

Thus f has no limit at infinity, although it is bounded. Boundedness gives absolute convergence for $s > 0$. At $s = 0$, the primitive contains the nonconstant periodic term $-\sin 2R/2 + (1 - \cos 2R)/4$, so it has no limit. For $s < 0$, the persistent sinusoid has positive fixed-length intervals with exponentially growing weighted integrals; the decaying time term cannot cancel it on late such intervals. Consequently the exact domain is $s > 0$.