

Question 3

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Recover the inverse of

$$F(s) = \frac{3s + 1}{s^2 + 4s + 13}.$$

Use the pairs for $e^{-at} \cos bt$ and $e^{-at} \sin bt$, whose numerators are $s + a$ and b , respectively, over $(s + a)^2 + b^2$.

Tasks

1. Complete the square and rewrite the numerator in terms of the shifted parameter.
2. Find and forward-check the inverse. Diagnose the candidate $e^{-2t}(3 \cos 3t + \frac{1}{3} \sin 3t)$.
3. Express the inverse as $Re^{-2t} \cos(3t + \phi)$ with $R > 0$ and $0 < \phi < \pi/2$. Find its first positive zero.
4. Give its exact real convergence interval and explain the roles of the decay envelope and oscillation frequency.

Question 3 – Solution

Strategy. Shift the numerator as well as the denominator; then distinguish phase, amplitude and exponential decay.

Step 1: Match the shifted pairs. The denominator is $(s+2)^2 + 9$, while $3s+1 = 3(s+2) - 5$. Therefore

$$F(s) = 3 \frac{s+2}{(s+2)^2 + 9} - \frac{5}{3} \frac{3}{(s+2)^2 + 9}.$$

Step 2: Invert and diagnose the sign error. The matching time function is

$$f(t) = e^{-2t} [3 \cos 3t - \frac{5}{3} \sin 3t].$$

Its forward transform reproduces $3(s+2) - 5 = 3s+1$. The proposed candidate instead has numerator $3(s+2) + 1 = 3s+7$. Completing the square without adjusting the numerator loses a constant contribution.

Step 3: Recover phase and the first zero. In $R \cos(3t + \phi) = R \cos \phi \cos 3t - R \sin \phi \sin 3t$, compare coefficients:

$$R = \frac{\sqrt{106}}{3}, \quad \phi = \arctan(5/9).$$

Both sine and cosine of ϕ are positive, so the specified quadrant removes any phase ambiguity. Starting at $f(0) = 3 > 0$, the first zero occurs when $3t + \phi = \pi/2$:

$$t_0 = \frac{\pi/2 - \arctan(5/9)}{3} = \frac{1}{3} \arctan(9/5).$$

Step 4: Separate envelope from oscillation. The bound $|f(t)| \leq R e^{-2t}$ gives absolute convergence for $s > -2$. At $s = -2$, the weighted signal is a nonzero sinusoid, whose primitive oscillates without a limit. For $s < -2$, positive fixed-length subintervals centered at successive cosine peaks have integrals growing exponentially, contradicting the Cauchy criterion. Thus $s > -2$ is exact. The decay rate is two and the angular frequency is three; neither gives the phase by itself. The graph shows both signed envelopes and the first zero.

