

Question 2

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Let

$$F(s) = \frac{s^2 + 3s + 4}{(s + 1)^3}.$$

You may use $\mathcal{L}\{t^n e^{-t}\} = n!/(s + 1)^{n+1}$ for nonnegative integers n and $s > -1$.

Tasks

1. Find the complete partial-fraction expansion at the repeated pole.
2. Invert every term, explaining the factorial factors. Verify the transform of your answer.
3. A student includes only a term $C/(s + 1)^3$. Show why this cannot represent F , and identify the missing time terms.
4. Compute $f(0)$, $f'(0)$, $f''(0)$ and the exact convergence interval. Check the three initial quantities against the first three large- s coefficients of F .

Question 2 – Solution

Strategy. A pole of order three allows all three reciprocal powers, and each power has its own factorial normalization.

Step 1: Expand around the pole. Set $p = s + 1$. Then $s^2 + 3s + 4 = p^2 + p + 2$, so

$$F(s) = \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3}.$$

Equivalently, multiplying by $(s+1)^3$ gives the polynomial identity $s^2 + 3s + 4 = (s+1)^2 + (s+1) + 2$.

Step 2: Apply the correctly normalized pairs. The inverse of $(s+1)^{-3}$ is $t^2 e^{-t}/2!$. Thus

$$\boxed{f(t) = e^{-t}(1 + t + t^2)}.$$

Its three forward transforms are $1/(s+1)$, $1/(s+1)^2$ and $2/(s+1)^3$, confirming the result. Forgetting $2!$ would double the quadratic term incorrectly.

Step 3: Explain the incomplete ansatz. An expression $C/(s+1)^3$ has constant numerator after multiplication by the denominator. It cannot equal the nonconstant polynomial $s^2 + 3s + 4$ for every s . The proposed inverse $Ct^2 e^{-t}/2$ lacks both e^{-t} and te^{-t} ; no choice of C can repair that omission.

Step 4: Check initial data and convergence. Differentiation gives

$$f' = e^{-t}(t - t^2), \quad f'' = e^{-t}(1 - 3t + t^2),$$

so $\boxed{f(0) = 1, f'(0) = 0, f''(0) = 1}$. Independently, expansion at large positive s gives

$$F(s) = \frac{1}{s} + \frac{0}{s^2} + \frac{1}{s^3} + O(s^{-4}).$$

These are the initial-value coefficients obtained by repeated integration by parts; this polynomial-exponential function satisfies all needed derivative bounds. Its defining integral converges absolutely for $s > -1$. At $s = -1$ the integrand is the positive polynomial $1 + t + t^2$, and for smaller s it grows still faster. The exact domain is $\boxed{s > -1}$.