

Question 1

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Consider

$$F(s) = \frac{s+5}{(s+1)(s+3)}.$$

The basic pair $\mathcal{L}\{e^{-at}\} = 1/(s+a)$ holds for $s > -a$.

Tasks

1. Find the inverse by partial fractions and verify the decomposition.
2. Determine the full real convergence interval of the recovered function.
3. Prove that the inverse is positive for all $t \geq 0$. Does its negative partial-fraction coefficient contradict positivity?
4. Find its initial value, its unique maximum and its limit at infinity. Explain why a sum of decaying modes need not decrease from the start.

Question 1 – Solution

Strategy. Resolve the two poles, then factor the resulting time expression before deciding its sign or shape.

Step 1: Resolve and invert. Writing $F = A/(s+1) + B/(s+3)$ gives $A+B=1$ and $3A+B=5$. Thus $A=2$, $B=-1$, and

$$f(t) = 2e^{-t} - e^{-3t}.$$

Transforming these two terms gives $2/(s+1) - 1/(s+3)$, whose numerator is $2(s+3) - (s+1) = s+5$. This verifies the entire rational expression, not just its pole locations.

Step 2: Retain the actual domain. For $s > -1$, both exponential integrals converge absolutely. Since $f = e^{-t}(2 - e^{-2t}) \geq e^{-t}$ on $t \geq 0$, its weighted integral diverges for $s \leq -1$. Hence the exact real domain is $s > -1$; other algebraic values of F are not ordinary transform values.

Step 3: Determine the sign. The factor $2 - e^{-2t}$ lies in $[1, 2)$, so $f(t) > 0$ everywhere. The negative coefficient belongs to one term in a decomposition; it does not determine the sign of the whole function. Here the faster-decaying negative term never dominates the positive term.

Step 4: Locate the peak. We have $f(0) = 1$, $f(t) \rightarrow 0$, and

$$f'(t) = e^{-t}(-2 + 3e^{-2t}).$$

The bracket decreases through zero once. Therefore f increases up to, then decreases after,

$$t_* = \frac{1}{2} \ln(3/2), \quad f(t_*) = \frac{4}{3} \sqrt{2/3}.$$

In particular $f'(0) = 1 > 0$: decay of each mode does not force monotonic decay of their signed combination. The graph marks the small but real overshoot.

