

Question 10

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Let f be twice continuously differentiable, with f, f', f'' of exponential order. Work at sufficiently large real s so that all transforms below converge and all exponential boundary terms vanish. You may use

$$\mathcal{L}\{f'\} = sF - f(0), \quad \mathcal{L}\{tf\} = -F', \quad \mathcal{L}\{f''\} = s^2F - sf(0) - f'(0).$$

Tasks

1. Compute $\mathcal{L}\{(tf)'\}$ and $\mathcal{L}\{tf'\}$ separately. Explain their difference using the product rule in time.
2. Verify both results explicitly for $f(t) = e^{-t}$.
3. Derive $\mathcal{L}\{t^2 f''\}$, carefully differentiating every s -dependent factor.
4. A student keeps only $s^2 F''$ in the last result. Identify the missing terms and verify the correct formula for $f = e^{-t}$ by transforming $t^2 e^{-t}$ directly.

Question 10 – Solution

Strategy. Multiplication by time and differentiation in time correspond to operations in s that must be applied in the correct order.

Step 1: Compare the two orders. Since $(tf)(0) = 0$, applying the time-derivative rule after time multiplication gives

$$\boxed{\mathcal{L}\{(tf)'\} = -sF'(s)}.$$

In the opposite order, parameter differentiation applies to the entire transform of f' :

$$\boxed{\mathcal{L}\{tf'\} = -\frac{d}{ds}[sF(s) - f(0)] = -F(s) - sF'(s)}.$$

Their difference is $F(s)$, exactly as required by $(tf)' - tf' = f$. The initial value $f(0)$ is constant in s and differentiates to zero.

Step 2: Check a concrete example. For $f = e^{-t}$, $F = 1/(s+1)$ and $F' = -1/(s+1)^2$. The formulas predict

$$\mathcal{L}\{(1-t)e^{-t}\} = \frac{s}{(s+1)^2}, \quad \mathcal{L}\{-te^{-t}\} = -\frac{1}{(s+1)^2}.$$

Direct use of the exponential and te^{-t} integrals verifies both, for $s > -1$. Their difference is $1/(s+1) = F$.

Step 3: Differentiate the complete expression twice. Multiplication by t^2 corresponds to two parameter derivatives with positive sign. Thus

$$\mathcal{L}\{t^2 f''\} = \frac{d^2}{ds^2}[s^2 F - sf(0) - f'(0)] = \boxed{2F + 4sF' + s^2 F''}.$$

The first derivative of $s^2 F$ is $2sF + s^2 F'$; differentiating again produces both product-rule contributions. The initial terms are at most linear in s , so their second derivatives vanish.

Step 4: Diagnose and verify. The proposed $s^2 F''$ omits $2F + 4sF'$. For $f = e^{-t}$, the correct result simplifies to

$$\frac{2}{s+1} - \frac{4s}{(s+1)^2} + \frac{2s^2}{(s+1)^3} = \boxed{\frac{2}{(s+1)^3}}.$$

Since $f'' = e^{-t}$, direct integration of $t^2 e^{-(s+1)t}$ gives this same value for $s > -1$. Differentiating only F while treating its prefactors as constants is the source of the error.