

Question 9

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

A nonnegative density on $t \geq 0$ has the form $f(t) = cte^{-at}$ with $a > 0$, $c > 0$ and $\int_0^\infty f(t) dt = 1$. Define its mean by $\mu = \int_0^\infty tf(t) dt$ and variance by $\sigma^2 = \int_0^\infty t^2 f(t) dt - \mu^2$. An exact measurement gives $\mu = 4$.

Tasks

1. Determine the relation between a, c from normalization and compute the transform.
2. Derive the first two moment identities using parameter differentiation. Recover a, c and the variance from the measured mean.
3. Could this same family have mean 4 and variance 4? Give an exact consistency relation for arbitrary positive mean.
4. Find the location of the density maximum for the recovered parameters and compare it with the mean. Explain why those locations differ.

Question 9 – Solution

Strategy. Normalization fixes one parameter, while derivatives of the transform at zero recover moments. A density's peak is not its mean.

Step 1: Normalize and transform. Integration by parts gives $\int_0^\infty te^{-at} dt = 1/a^2$. Thus $c = a^2$, and exponential shifting yields

$$F(s) = \frac{a^2}{(s+a)^2}, \quad s > -a.$$

The nonnegative polynomial-exponential tail diverges at or below $-a$. In particular, $F(0) = 1$, as normalization requires.

Step 2: Extract moments and parameters. Since zero lies strictly inside the convergence interval, exponential decay justifies differentiation near zero:

$$-F'(0) = \int_0^\infty tf(t) dt, \quad F''(0) = \int_0^\infty t^2 f(t) dt.$$

The rational expression gives $F'(s) = -2a^2/(s+a)^3$ and $F''(s) = 6a^2/(s+a)^4$. Hence

$$\mu = \frac{2}{a}, \quad \sigma^2 = \frac{6}{a^2} - \frac{4}{a^2} = \frac{2}{a^2}.$$

From $\mu = 4$, we obtain

$$a = \frac{1}{2}, \quad c = \frac{1}{4}, \quad \sigma^2 = 8, \quad F(s) = \frac{1}{(2s+1)^2}.$$

Step 3: Test consistency rather than refitting. Eliminating a gives $\sigma^2 = \mu^2/2$. Thus mean four forces variance eight, and the pair $(4, 4)$ is incompatible with this family. Changing c cannot repair it because normalization already fixes $c = a^2$.

Step 4: Locate the peak. For general a , $f'(t) = a^2e^{-at}(1-at)$, positive before $t = 1/a$ and negative afterward. The unique density maximum is therefore at $t = 1/a = 2$, while the mean is four. The long right tail contributes to the mean; the point of largest density measures a different feature. The graph marks both the mode and the mean for $f(t) = te^{-t/2}/4$.

