

Question 8

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Suppose f is continuous, bounded and nonnegative on $[0, \infty)$. Consider the candidate expression

$$R(s) = \frac{s}{(s+1)^2}, \quad s > 0.$$

Tasks

1. Derive the sign requirements on $F(s)$, $F'(s)$ and $F''(s)$ imposed by these hypotheses. State when $F(s) > 0$.
2. Although $R(s) > 0$ for every $s > 0$, prove that it cannot be the transform of such a nonnegative f .
3. Construct a continuous bounded sign-changing function g whose transform is R , using basic exponential pairs. Verify its sign change and full real convergence interval.
4. Compute $\int_0^\infty g(t) dt$ and explain why positivity of a transform throughout $s > 0$ does not imply positivity of its original function.

Question 8 – Solution

Strategy. Positivity of the transform is only one consequence of a nonnegative signal; parameter derivatives supply stronger necessary tests.

Step 1: Derive the necessary signs. If $0 \leq f \leq M$, then the kernels obtained by differentiating once or twice are dominated locally in $s > 0$ by constant multiples of $te^{-s_0 t/2}$ and $t^2 e^{-s_0 t/2}$. Differentiation under the integral is therefore justified, and

$$F(s) \geq 0, \quad F'(s) = - \int_0^\infty te^{-st} f(t) dt \leq 0,$$

$$F''(s) = \int_0^\infty t^2 e^{-st} f(t) dt \geq 0.$$

If f is not identically zero, continuity gives a positive interval contribution and hence $F(s) > 0$ for every $s > 0$.

Step 2: Test the positive candidate. Although $R(s) > 0$,

$$R'(s) = \frac{1-s}{(s+1)^3} > 0 \quad (0 < s < 1).$$

This contradicts the required nonpositive derivative. Thus no continuous bounded nonnegative signal can have this transform.

Step 3: Find a valid signed signal. Rewrite $R = 1/(s+1) - 1/(s+1)^2$. The basic exponential integrals give

$$g(t) = e^{-t}(1-t), \quad \mathcal{L}\{g\}(s) = R(s), \quad s > -1.$$

The function is positive for $0 \leq t < 1$, zero at one, and negative for $t > 1$. It is continuous and bounded. At or below $s = -1$, its eventually negative polynomial or growing tail has no finite improper integral, so the domain is exact.

Step 4: Explain the weighted cancellation. At $s = 0$, the integral exists and equals $R(0) = 0$, also verified by $\int e^{-t} = 1$ and $\int te^{-t} = 1$. For positive s , the extra weight suppresses the late negative portion more strongly, producing a positive transform. Consequently $F > 0$ alone is insufficient to infer $f \geq 0$. The two plots show a sign-changing signal and its everywhere-positive transform on $s > 0$.

